

Covariant Quantum Gravity with Continuous Quantum Geometry I: Covariant Hamiltonian Framework

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(Dated: 27 September 2016)

The first part of the series is devoted to the formulation of the Einstein-Cartan Theory within the covariant hamiltonian framework. In the first section the general multisymplectic approach is revised and the notion of the d-jet bundles is introduced. Since the whole Standard Model Lagrangian (including gravity) can be written as the functional of the forms, the structure of the d-jet bundles is more appropriate for the covariant hamiltonian analysis than the standard jet bundle approach. The definition of the local covariant Poisson bracket on the space of covariant observables is recalled. The main goal of the work is to show that the gauge group of the Einstein-Cartan theory is given by the semidirect product of the local Lorentz group and the group of spacetime diffeomorphisms. Vanishing of the integral generators of the gauge group is equivalent to equations of motion of the Einstein-Cartan theory and the local covariant algebra generated by Noether's currents is closed Lie algebra.

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I. INTRODUCTION

The first part of the series, which proposes as hypothesis a new theory of Covariant Quantum Gravity (CQG) with continuous quantum geometry, is dealing with a formulation of the Einstein-Cartan Theory within the covariant hamiltonian framework. The Einstein-Cartan theory, also familiar as the Kibble-Sciama theory, is a gauge theory where the local Poincaré group plays a role of the gauge symmetry¹⁻³. Standard ADM formulation⁴ of General Relativity requires time+space splitting of the spacetime therefore the Hamilton formalism, which is necessary for any rigorous quantum formulation, breaks the explicit covariance and the algebra of constraints is no more closed Lie algebra, similar result can be obtained within the Einstein-Cartan theory⁵. In the case when the gravitation is interacting with the pressure-free dust then there exists the privileged system of the coordinates comoving with every single grain of the dust which enables to rewrite equivalently the ADM constraints in such a way that they form closed Lie algebra⁶. Kuchař also tried to rewrite the ADM constraints for the gravitation interacting with scalar field but in this case the ADM constraints can be rewritten only implicitly and the result depends on the solution of the equations of motion⁷. Another possible solution of this Lie algebra problem was proposed in the Phoenix Project where all constraints of the system are contained in a single Master Constraint⁸, but the Master Constraint for LQG is quadratic in the Hamiltonian constraint and given Lie algebra is no more associated with the local Poincaré group.

The problem just mentioned yields the question whether there exists covariant hamiltonian formalism which can be applied here and which does not require the space+time splitting. Fortunately, we know that there exists an affirmative answer based on ideas of the multisymplectic geometry which generalizes familiar symplectic structures⁹⁻¹².

Usually, the covariant hamiltonian description works with a notion of the jet bundles. The jet bundle is a fibre bundle constructed from the given configuration bundle \mathbf{Y} over the spacetime \mathbf{M} , with the local coordinates (x^μ, y^A) , by adding the first derivatives of the variables y^A to \mathbf{Y} , i.e. locally (x^μ, y^A, v_μ^A) . The Standard Model Lagrangian (including gravity) can be written as a functional of the forms over the spacetime \mathbf{M} with values in a certain vector space (or in its submanifold as in the case of gravity) and their gauge covariant exterior derivatives, therefore it is more suitable to work within this structure, called the d-jet bundles, instead of the jet bundles. Therefore the basic results (covariant hamiltonian equations of motion, momentum maps, Noether's charges) in the language of the d-jet bundles should be introduced. This the task of the section II.

The section III is devoted to the construction of the local covariant Poisson bracket. We recall basic definitions of the local covariant bracket, observables and associated hamiltonian vector fields¹³ for the general multisymplectic manifold. Since we introduce two multisymplectic structures, the kinematical and dynamical, in the section II, we need to know how these two local covariant brackets are related. We also need to explore the general shape of the local observable, which is given by sum of generators of the group $\mathfrak{Diff}(\mathbf{M})$ of all spacetime diffeomorphisms and \mathbf{M} -horizontal simply differentiable $(n-1)$ -forms, see theorem III.1. Thus on general level we arrive into Lie algebra of local covariant observables. But as we know, the standard quantization procedure is based on searching of representations of different kind of, an integral-like, observables and Poisson bracket. We do not proceed such construction here, but rather we left it to the second part of the series where we explore instantaneous formalism¹⁴ in more detail. On the other hand, the searching of the local Poisson algebra representation is also considered and explored in the literature and this step is used to be called a pre-quantization^{15,16}, but in some sense it can be viewed as the first quantization.

We deal with the Einstein-Cartan theory in the section IV. At first we introduce a graded bun-

dle of the right-handed coframes whose elements are interpreted as orthonormal vierbeins in the Einstein-Cartan theory. As another independent variable of the Einstein-Cartan theory is considered a metric-compatible connection and we finally arrive at the full covariant configuration bundle and the multisymplectic structure over it. Since the covariant Legendre map is singular we must proceed a multisymplectic reduction. Next, we explore equations of motion in the point of view of the covariant hamiltonian formalism. As the last thing, we find out that the gauge group of the Einstein-Cartan theory is given by the semidirect product of the local Lorentz group $SO(\boldsymbol{\eta}, \mathbf{M})$ and the group of spacetime diffeomorphisms $\mathfrak{Diff}(\mathbf{M})$ and show that equations of motion are given by vanishing of Noether's charges related to generators of the gauge group.

NOTATION:

We use following convention in the series.

\mathbf{M} – Spacetime manifold with $\dim \mathbf{M} = n$,
 Σ – Spatial manifold with $\dim \Sigma = n - 1$.

Spacetime indices are labeled by $\mu, \nu, \bar{\mu}, \hat{\mu}, \dots = 0, 1, \dots, n - 1$, where indices with hats, bars are considered as standard without any additional meaning. Spatial indices are labeled by $\alpha, \beta, \gamma, \bar{\alpha}, \hat{\alpha}, \dots = 1, \dots, n - 1$.

Multi-index notation

Let $B_{\mu_{p+1}\dots\mu_q}$ be totally antisymmetric ($0 \leq p < q \leq n$) then we set

$$B_{(\mu)_q}^p = B_{\mu_{p+1}\dots\mu_q},$$

$$B_{(\mu)_q} = B_{(\mu)_q}^0,$$

$$B_{(\mu)^p} = B_{(\mu)_n}^p,$$

$$B_{(\mu)} = B_{(\mu)_n}^0.$$

Summation in multi-index notation

$$B_{(\mu)_q}^p C^{(\mu)_q^p} = \frac{1}{(q-p)!} B_{\mu_{p+1}\dots\mu_q} C^{\mu_{p+1}\dots\mu_q}.$$

Lebesgue's coordinate measures

$$\mathbf{d}x^{(\mu)_q^p} = \mathbf{d}x^{\mu_{p+1}} \wedge \dots \mathbf{d}x^{\mu_q},$$

$$\mathbf{d}\Sigma_{(\mu)_q} = \varepsilon_{(\mu)_q(\mu)_q} \mathbf{d}x^{(\mu)_q},$$

where $\varepsilon_{(\mu)}$ and $\bar{\varepsilon}_{(\mu)}$ are Levi-Civita symbols ($\varepsilon_{01\dots(n-1)} = \bar{\varepsilon}^{01\dots(n-1)} = 1$).

Coframe indices runs through $a, b, \bar{a}, \dots = 0, 1, \dots, n - 1$ and Levi-Civita symbols $\varepsilon_{(a)}$ and $\bar{\varepsilon}^{(a)}$ are given by ($\varepsilon_{01\dots(n-1)} = \bar{\varepsilon}^{01\dots(n-1)} = 1$) and Minkowski metric tensor has signature $(\eta_{ab}) = \text{diag}(-1, +1, +1, \dots, +1, +1)$. General multi-indices A, B, \bar{A}, \dots are running through some certain finite set depending on considered theory.

II. COVARIANT HAMILTONIAN FORMALISM: D-JET BUNDLES, MULTISYMPLECTIC MANIFOLDS, COVARIANT MOMENTUM MAPS AND EQUATIONS OF MOTION

In the classical mechanics the hamiltonian analysis takes place on the symplectic manifold $(\mathbf{P} = T^*\mathbf{Y}, \omega)$, where \mathbf{Y} is the finite dimensional configuration space and $\omega = -\mathbf{d}\vartheta = -\mathbf{d}(p_A \mathbf{d}y^A)$ is the canonical symplectic 2-form on \mathbf{P} . In the case of the classical field theory one usually starts with the infinite dimensional configuration manifold and then canonically constructs the infinite dimensional phase space. This construction requires the time+space splitting of the variables and the Legendre map relates the velocities, i.e. time derivatives of variables, with the canonical momenta. This breaks an explicit covariance. There also exists another approach⁹⁻¹¹ which works with the multisymplectic structure and even more these two constructions are mutually complementary as we will see in the next part of the series. In this approach the infinite-dimensional configuration (or phase) space is replaced by the set of all (sufficiently smooth) sections of a certain finite-dimensional fibre bundle over the spacetime. While in the symplectic case the Legendre map gives, in the non-degenerate case, one-to-one relation between the velocities and the canonical momenta, the multisymplectic canonical momenta are related by the generalized or covariant Legendre map with the exterior derivatives of the fields.

Let \mathbf{M} be a spacetime manifold with dimension n with local coordinates be x^μ , where $\mu, \nu, \dots \in \{0, 1, \dots, n-1\}$. Since we are interested in gravity where the metric is one of the observables it is also assumed that \mathbf{M} is the topological smooth manifold only. It is well known^{4,17} that the equations of motion of the standard fields are well possessed if there exists a global Cauchy surface and spacetime has a structure $\mathbf{M} = \mathbb{R} \times \Sigma$, where $x^0 \in \mathbb{R}$ is interpreted as the time and $\Sigma_t = \{\mathbf{x} \in \mathbf{M}; x^0(\mathbf{x}) = t\}$ are supposed to be the achronal sections through \mathbf{M} and play role of the Cauchy surfaces. As it was mentioned in the case of gravity there is no background metric and therefore the notion of the Cauchy surface depends on the solution. But if one assumes only the product condition $\mathbf{M} = \mathbb{R} \times \Sigma$ and if for the initial conditions the initial embedding $\Sigma_{t_{\text{ini}}}$ of Σ is the Cauchy surface then the equations of motion for the gravitational field are well possessed^{4,17}. Thus, let $\mathbf{M} = \mathbb{R} \times \Sigma$. In order to avoid an analysis of the boundary terms and the overlapping conditions it is also assumed that Σ is compact boundaryless orientable $(n-1)$ -dimensional manifold and the considered fields are globally defined, therefore the bundles constructed in this part are trivial. Nonetheless, we should discuss the overlapping conditions later in the case of Einstein-Cartan theory since in the case of general Σ an orthonormal coframe field needn't be defined globally and we will do this in the next part of the series where the instantaneous formalism is considered. But if we do not talk about sections then lot of the results achieved here are valid also in nontrivial case by restriction of smearing forms, vectors having contractible supports on \mathbf{M} .

The standard multisymplectic methods work with "scalar" fields y^A and their spacetime derivatives $y^A_{,\mu}$. This means that if one wants to describe for example an electromagnetic field then $y^A \equiv A_\mu$ and the coordinate index μ is hidden in the general multi-index A of y^A . On the other hand the Einstein-Hilbert-Palatini and the Standard model Lagrangians can be formulated within forms (0-forms: scalar, Dirac, neutrino fields or 1-forms: electroweak, strong, gravitational connections or orthonormal coframe field) and their gauge-covariant exterior derivatives. Hence, it seems to be more suitable to develop the multisymplectic methods using directly the structures of the exterior algebra over the spacetime \mathbf{M} .

By graded manifold \mathbf{Y} we mean following (we are introducing the simplified definition rather than its general version in terms of categories).

Definition II.1. Let $\mathbb{p}_{\mathbf{M}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{M}$ be a fibre bundle, where \mathbf{M}, \mathbf{Y} are open finite dimensional smooth

manifolds and the bundle projection $\mathbb{p}_{\mathbf{M}, \mathbf{Y}}$ is smooth map. If there exists a vector subbundle $\mathbf{M}\mathbf{V}$ of the vector bundle $\Lambda(\mathbf{M}, \mathbf{V})$ of forms over \mathbf{M} with values in the real finite dimensional vector space \mathbf{V} such that the typical fibre \mathbf{Y}^f of the bundle \mathbf{Y} is submanifold of the typical fibre vector space $\mathbf{M}\mathbf{V}^f$ of the bundle $\mathbf{M}\mathbf{V}$ and if equality $\mathbf{T}\mathbf{Y}^f = \mathbf{T}_{\mathbf{Y}^f}\mathbf{M}\mathbf{V}^f$ holds then \mathbf{Y} is called graded manifold.

NOTE: \mathbf{Y} might be also considered as complex manifold but for $\forall \mathbf{y} \in \mathbf{Y}$ we require $\mathbf{y}^* \in \mathbf{Y}_{\mathbf{x}}$, where $\mathbf{x} = \mathbb{p}_{\mathbf{M}, \mathbf{Y}}(\mathbf{y})$ and $\mathbf{Y}_{\mathbf{x}} = \mathbb{p}_{\mathbf{M}, \mathbf{Y}}^{-1}(\mathbf{x})$ is the fibre over \mathbf{x} containing \mathbf{y} and $*$ is a complex conjugation inherited from the complex vector space \mathbf{V} .

The coordinates \mathbf{y}^A on the fibre $\mathbf{Y}_{\mathbf{x}}$ are forms in \mathbf{x} with values in a certain real vector space (or its submanifold as in the case of gravity) labeled by the index A which is independent on the \mathbf{M} -coordinate indices μ, ν, \dots . In general, the degree q of \mathbf{y}^A may depends on A , but if there is no confusion we do not write this dependence explicitly, i.e.

$$\mathbf{y}^A = \frac{1}{q!} y_{\mu_1 \dots \mu_q}^A \mathbf{d}x^{\mu_1} \wedge \dots \wedge \mathbf{d}x^{\mu_q} = y_{(\mu)_q}^A \mathbf{d}x^{(\mu)_q}$$

keeping in mind this assumption.

Graded manifold \mathbf{Y} plays role of the basic playground where all possible physical configurations or states are given by the set of all sections, i.e. maps $\varphi : \mathbf{M} \rightarrow \mathbf{Y}$. We denote the set of such sections by $\text{Sec}(\mathbf{M}, \mathbf{Y})$. Section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ defines a submanifold $\varphi(\mathbf{M}) \subset \mathbf{Y}$ given locally by $(x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x}))$. Two sections $\varphi, \varphi' \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ are equivalent $\varphi \stackrel{\pm}{\sim} \varphi'$ in the point $\mathbf{x} \in \mathbf{M}$ if $\varphi(\mathbf{x}) = \varphi'(\mathbf{x})$, i.e. $(x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x})) = (x^\mu(\mathbf{x}), \mathbf{y}_{\varphi'}^A(\mathbf{x}))$, and $\mathbf{d}\mathbf{y}_\varphi^A(\mathbf{x}) = \mathbf{d}\mathbf{y}_{\varphi'}^A(\mathbf{x})$, where \mathbf{d} is the exterior derivative on \mathbf{M} . If the equivalence class of $\varphi(\mathbf{x})$ is denoted $[\varphi]_{\mathbf{x}}$ then the set defined by

$$\mathbf{JY} := \bigcup_{\substack{\mathbf{x} \in \mathbf{M} \\ \varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})}} [\varphi]_{\mathbf{x}}$$

is the manifold which can be locally coordanized by $(x^\mu, \mathbf{y}^A, \mathbf{v}^A)$ where \mathbf{v}^A , the generalized velocities of \mathbf{y}^A , are $(q+1)$ -forms on \mathbf{M} . The projection $\mathbb{p}_{\mathbf{Y}, \mathbf{JY}} : \mathbf{JY} \rightarrow \mathbf{Y}$, given by $\mathbb{p}_{\mathbf{Y}, \mathbf{JY}} : (x^\mu, \mathbf{y}^A, \mathbf{v}^A) \mapsto (x^\mu, \mathbf{y}^A)$, turns \mathbf{JY} into the fibre bundle and chain $\mathbf{JY} \xrightarrow{\mathbb{p}_{\mathbf{Y}, \mathbf{JY}}} \mathbf{Y} \xrightarrow{\mathbb{p}_{\mathbf{M}, \mathbf{Y}}} \mathbf{M}$ is called d-jet triple. Let $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ then $\mathbf{j}\varphi : \mathbf{M} \rightarrow \mathbf{JY}$ defined by $\mathbf{j}\varphi : \mathbf{x} \mapsto (x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x}), \mathbf{d}\mathbf{y}_\varphi^A(\mathbf{x}))$ is a section on the bundle $\mathbb{p}_{\mathbf{M}, \mathbf{Y}} \circ \mathbb{p}_{\mathbf{Y}, \mathbf{JY}} = \mathbb{p}_{\mathbf{M}, \mathbf{JY}} : \mathbf{JY} \rightarrow \mathbf{M}$ called the d-jet prolongation of φ . Section of the bundle $\mathbb{p}_{\mathbf{M}, \mathbf{JY}} : \mathbf{JY} \rightarrow \mathbf{M}$ is called holonomic if it is d-jet prolongation of some section of the bundle $\mathbb{p}_{\mathbf{M}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{M}$. If for $\forall A$ we have $q = 0$ then d-jet bundle reduces into the the first jet bundle. For $q = n$ the d-jet bundle is trivial, since every \mathbf{M} horizontal $(n+1)$ -form on \mathbf{M} is trivially vanishing, thus $\mathbf{JY} \simeq \mathbf{Y}$.

Let $\mathbf{JY} \rightarrow \mathbf{Y} \rightarrow \mathbf{M}$ and $\mathbf{JZ} \rightarrow \mathbf{Z} \rightarrow \mathbf{N}$ be two d-jet triples and let there exists a homomorphism between bundles $\mathbf{Y} \rightarrow \mathbf{M}$ and $\mathbf{Z} \rightarrow \mathbf{N}$, i.e. two diffeomorphisms $\eta_{\mathbf{NM}} : \mathbf{M} \rightarrow \mathbf{N}$ and $\eta_{\mathbf{ZY}} : \mathbf{Y} \rightarrow \mathbf{Z}$ tied by $\eta_{\mathbf{NM}} \circ \mathbb{p}_{\mathbf{M}, \mathbf{Y}} = \mathbb{p}_{\mathbf{N}, \mathbf{Z}} \circ \eta_{\mathbf{ZY}}$. If $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})$, then $\eta_{\mathbf{ZY}} \circ \varphi \circ (\eta_{\mathbf{NM}})^{-1}$ is a section on $\mathbf{Z} \rightarrow \mathbf{N}$. If for every point $\forall \mathbf{x} \in \mathbf{M}$ and for arbitrary two equivalent sections $\varphi \stackrel{\pm}{\sim} \varphi' \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ the structural condition

$$\eta_{\mathbf{ZY}} \circ \varphi \circ (\eta_{\mathbf{NM}})^{-1} \stackrel{\eta_{\mathbf{NM}}(\mathbf{x})}{\sim} \eta_{\mathbf{ZY}} \circ \varphi' \circ (\eta_{\mathbf{NM}})^{-1} \quad (1)$$

is satisfied, then it is possible to prolong the map $\eta_{\mathbf{ZY}}$ to the d-jet part of the triples by

$$\eta_{\mathbf{JZY}} \equiv \mathbf{j}\eta_{\mathbf{ZY}} := [\varphi]_{\mathbf{x}} \mapsto [\eta_{\mathbf{ZY}} \circ \varphi \circ (\eta_{\mathbf{NM}})^{-1}]_{\eta_{\mathbf{NM}}(\mathbf{x})}.$$

A tri-map $\eta = (\eta_{\mathbf{NM}}, \eta_{\mathbf{ZY}}, \mathbf{j}\eta_{\mathbf{ZY}}) : (\mathbf{M}, \mathbf{Y}, \mathbf{JY}) \rightarrow (\mathbf{N}, \mathbf{Z}, \mathbf{JZ})$ is called d-jet homomorphism. We write $\eta \in \mathfrak{Hom}(\mathbf{Y}, \mathbf{Z})$, or simply $\eta_{\mathbf{ZY}} \in \mathfrak{Hom}(\mathbf{Y}, \mathbf{Z})$. In the case when two d-jet triples coincide then the tri-map

η is called d-jet automorphism and set of all d-jet automorphisms is denoted by $\mathfrak{Aut}(\mathbf{Y})$. In general it may happen that the inverse maps $(\eta_{\mathbf{N},\mathbf{M}}^{-1}, \eta_{\mathbf{Z},\mathbf{Y}}^{-1})$ can not be prolonged into d-jet homomorphism, see example given by transformations (45) and (46). In the case when it is possible to prolong the inverse maps $(\eta_{\mathbf{N},\mathbf{M}}^{-1}, \eta_{\mathbf{Z},\mathbf{Y}}^{-1})$ to d-jet homomorphism then the tri-map $\eta = (\eta_{\mathbf{NM}}, \eta_{\mathbf{ZY}}, j\eta_{\mathbf{ZY}})$ is called d-jet diffeomorphism and set of all d-jet diffeomorphisms is denoted by $\mathbf{d}\text{-}\mathfrak{Diff}(\mathbf{Y})$.

Let \mathbf{F} be general (sufficiently smooth) function $\mathbf{F} : \Lambda^m \mathbf{M} \rightarrow \Lambda^m \mathbf{M}$ depending on variable $\mathbf{v} \in \Lambda^m \mathbf{M}$. We say that \mathbf{F} is simply differentiable if its Taylor expansion to the first order can be written as

$$\mathbf{F}(\mathbf{v} + \delta\mathbf{v}) = \mathbf{F}(\mathbf{v}) + \delta\mathbf{v} \wedge \frac{\partial^L \mathbf{F}}{\partial \mathbf{v}} = \mathbf{F}(\mathbf{v}) + \frac{\partial^R \mathbf{F}}{\partial \mathbf{v}} \wedge \delta\mathbf{v}. \quad (2)$$

$(m-r)$ -forms $\frac{\partial^L \mathbf{F}}{\partial \mathbf{v}}$ or $\frac{\partial^R \mathbf{F}}{\partial \mathbf{v}}$ are called left or right simple derivatives of \mathbf{F} with respect to \mathbf{v} , respectively. These forms are related by

$$\frac{\partial^R \mathbf{F}}{\partial \mathbf{v}} = (-1)^{r(m-r)} \frac{\partial^L \mathbf{F}}{\partial \mathbf{v}}.$$

In special case $m = n$ we have

$$\mathbf{F}(\mathbf{v} + \delta\mathbf{v}) = \tilde{F}(v_{(\mu)_r} + \delta v_{(\mu)_r}) \mathbf{d}\Sigma = \tilde{F} \mathbf{d}\Sigma + \frac{\partial \tilde{F}}{\partial v_{(\mu)_r}} \delta v_{(\mu)_r} \mathbf{d}\Sigma,$$

where $\mathbf{d}\Sigma$ is coordinate Lebesgue's volume form. Since in general

$$\delta v_{(\mu)_r} \mathbf{d}\Sigma = \delta\mathbf{v} \wedge \mathbf{d}\Sigma_{(\mu)_r} \quad (3)$$

it is obvious that such \mathbf{F} is always simply differentiable.

Let $\eta = (\eta_{\mathbf{NM}}, \eta_{\mathbf{ZY}}, j\eta_{\mathbf{ZY}}) \in \mathfrak{Hom}(\mathbf{Y}, \mathbf{Z})$ be d-jet homomorphism from \mathbf{Y} to \mathbf{Z} then by definition $\eta_{\mathbf{ZY}}$ is diffeomorphism, i.e. infinitely many times differentiable map, but it does not mean that $\eta_{\mathbf{ZY}}$ is (in)finitely many times simply differentiable. Following theorem says how general d-jet homomorphism can be characterised within the notion of the simple differentiation.

Theorem II.1. *Bundle homomorphism $(\eta_{\mathbf{NM}}, \eta_{\mathbf{ZY}})$ can be extended into d-jet homomorphism if and only if*

$$\begin{aligned} \eta_{\mathbf{NM}} : (x^\mu) &\mapsto ((\eta_{\mathbf{NM}})^{\bar{\mu}}(x^\mu)), \\ \eta_{\mathbf{ZY}} : (x^\mu, \mathbf{y}^A) &\mapsto ((\eta_{\mathbf{NM}})^{\bar{\mu}}(x^\mu), (\eta_{\mathbf{NM}}^{-1})^* \bar{\mathbf{Y}}^{\bar{A}}(x^\mu, \mathbf{y}^A)), \end{aligned}$$

where each $\bar{\mathbf{Y}}^{\bar{A}} \in \text{Sec}(\mathbf{Y}, \Lambda \mathbf{Y})$ is simply differentiable \mathbf{M} -horizontal form on \mathbf{Y} .

Proof. Let $(x^\mu, \mathbf{y}^A = y_{(\mu)_q}^A \mathbf{d}x^{(\mu)_q})$ or $(\bar{x}^{\bar{\mu}}, \bar{\mathbf{y}}^{\bar{A}} = \bar{y}_{(\bar{\mu})_{\bar{q}}}^{\bar{A}} \mathbf{d}\bar{x}^{(\bar{\mu})_{\bar{q}}})$ be local coordinates on bundle $\mathbf{Y} \rightarrow \mathbf{M}$ or $\mathbf{Z} \rightarrow \mathbf{N}$, respectively, then $(\eta_{\mathbf{NM}}, \eta_{\mathbf{ZY}})$ is bundle homomorphism if and only if it looks like

$$\begin{aligned} \eta_{\mathbf{NM}} : (x^\mu) &\mapsto ((\eta_{\mathbf{NM}})^{\bar{\mu}}(x^\mu)), \\ \eta_{\mathbf{ZY}} : (x^\mu, y_{(\mu)_q}^A) &\mapsto ((\eta_{\mathbf{NM}})^{\bar{\mu}}(x^\mu), \mathbf{Z}^{\bar{A}}(x^\mu, y_{(\mu)_q}^A)). \end{aligned}$$

Now, let $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ be arbitrary section given locally $\varphi(\mathbf{x}) = (x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x})) \equiv (x^\mu(\mathbf{x}), y_{(\mu)_q}^A(x^\mu(\mathbf{x})))$, then section on $\mathbf{Z} \rightarrow \mathbf{N}$ is given by

$$\begin{aligned} \eta_{\mathbf{ZY}} \circ \varphi \circ (\eta_{\mathbf{NM}})^{-1}(\bar{\mathbf{x}}) &= \left(\bar{x}^{\bar{\mu}}(\bar{\mathbf{x}}), \mathbf{Z}_{(\bar{\mu})_{\bar{q}}}^{\bar{A}}(x^\mu(\bar{\mathbf{x}}), y_{(\mu)_q}^A(x^\mu(\bar{\mathbf{x}}))) \right), \\ &= (\bar{x}^{\bar{\mu}}(\bar{\mathbf{x}}), \mathbf{Z}_\varphi^{\bar{A}}(\bar{\mathbf{x}})), \end{aligned}$$

where $x^\mu(\bar{\mathbf{x}}) = (\eta_{\mathbf{NM}}^{-1})^\mu(\bar{\mathbf{x}})$ for short. Now if we set $\bar{\mathbf{Y}}^{\bar{A}} = (\eta_{\mathbf{NM}})^* \mathbf{Z}^{\bar{A}}$ and calculate $\mathbf{dZ}_\varphi^{\bar{A}}$ then we get

$$(\eta_{\mathbf{NM}})^* \mathbf{dZ}_\varphi^{\bar{A}} = \mathbf{d}\bar{\mathbf{Y}}_\varphi^{\bar{A}} = \bar{Y}_{(\lambda)\bar{q},\nu}^{\bar{A}} \mathbf{d}x^\nu \wedge \mathbf{d}x^{(\lambda)\bar{q}} + \frac{\partial \bar{Y}_{(\lambda)\bar{q}}^{\bar{A}}}{\partial y_{(\mu)q}^A} y_{(\mu)q,\nu}^A \mathbf{d}x^\nu \wedge \mathbf{d}x^{(\lambda)\bar{q}}.$$

Structural condition (1) yields that $\mathbf{dZ}_\varphi^{\bar{A}}$ should depends only on antisymmetric part of $y_{(\mu)q,\nu}^A$. Since index ν is summarized with $\mathbf{d}x^\nu$ we have that last term should looks like

$$\frac{\partial \bar{Y}_{(\lambda)\bar{q}}^{\bar{A}}}{\partial y_{(\mu)q}^A} y_{(\mu)q,\nu}^A \mathbf{d}x^\nu \wedge \mathbf{d}x^{(\lambda)\bar{q}} = y_{(\mu)q,\nu}^A \mathbf{d}x^\nu \wedge \mathbf{d}x^{(\mu)q} \wedge \mathcal{Q}_{A(\lambda)\bar{q}}^{\bar{A}} \mathbf{d}x^{A(\lambda)\bar{q}}.$$

This condition is satisfied if only if $q \leq \bar{q}$ and

$$\frac{\partial \bar{Y}_{(\lambda)\bar{q}}^{\bar{A}}}{\partial y_{(\mu)q}^A} = \binom{\bar{q}}{q} \delta_{[(\lambda)q]}^{(\mu)q} \mathcal{Q}_{A(\lambda)\bar{q}}^{\bar{A}}$$

or $\frac{\partial \bar{Y}_{(\lambda)\bar{q}}^{\bar{A}}}{\partial y_{(\mu)q}^A} = 0$. Using this formula in Taylor expansion of $\bar{\mathbf{Y}}^{\bar{A}}$ yields immediately that $\bar{\mathbf{Y}}^{\bar{A}}$ is simply differentiable. Proof of opposite implication is straightforward, hence we leave it for reader. \square

Now, we want to introduce a dual of the d-jet bundle which plays an important role in the covariant hamiltonian formalism. It can be seen that \mathbf{v}^A -part of coordinates on the d-jet bundle transforms for general d-jet diffeomorphism in an affine way. In other words \mathbf{v}^A behaves as point in affine space rather than vector. So, if one wants to talk about dual of \mathbf{v}^A -space then one must consider its affine dual. For the purposes of the covariant hamiltonian formalism it is the most suitable to consider affine maps to the space $\Lambda^n \mathbf{M}$ of n -forms on \mathbf{M} . Using (3) we see that these can be labeled by $(n - q - 1)$ -forms \mathbf{p}_A and n -form \mathbf{h} where the affine map α is given by

$$\alpha(\mathbf{v}^A) = (-1)^{n-q} \mathbf{p}_A \wedge \mathbf{v}^A + \mathbf{h}.$$

The factor $(-1)^{n-q}$ is chosen in such a way that the local covariant Poisson bracket constructed in the next section of canonically conjugated coordinates $\mathbf{y}^A, \mathbf{p}_B$ is equal to identity, see lemma III.4. So, the dual \mathbf{JY}^* of the d-jet bundle \mathbf{JY} is defined as a fibre bundle $\mathbb{p}_{\mathbf{Y}, \mathbf{JY}^*} : \mathbf{JY}^* \rightarrow \mathbf{Y}$ locally described by coordinates as $(x^\mu, \mathbf{y}^A, \mathbf{p}_A, \mathbf{h})$.

There can be constructed a couple of forms on the d-jet dual \mathbf{JY}^* . The first canonical n -forms, called the kinematical Cartan-Poincaré form, is defined as

$$\theta = (-1)^{n-q} \mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A + \mathbf{h} \quad (4)$$

and the second canonical $(n + 1)$ -form, called the kinematical multisymplectic form, is given by

$$\omega = -\mathbf{d}\theta = (-1)^{n-q-1} \mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A - \mathbf{d}\mathbf{h}. \quad (5)$$

In mechanics the Legendre transformation relates the velocities with the canonical momenta and the Lagrangian with the Hamiltonian. Such transformation is driven by the Lagrange function. In the multisymplectic context it is similar. Lagrange function $\mathbf{L} : (x^\mu, \mathbf{y}^A, \mathbf{v}^A) \mapsto \mathbf{L}(x^\mu, \mathbf{y}^A, \mathbf{v}^A) \in \Lambda^n \mathbf{M}$ is a map from the d-jet bundle \mathbf{JY} to the space of n -forms on \mathbf{M} . Let $\gamma, \gamma_0 \in \mathbf{JY}_\mathbf{y} := (\mathbb{p}_{\mathbf{Y}, \mathbf{JY}})^{-1}(\mathbf{y})$ then the affine approximation of the Lagrangian in the point γ_0 on $\mathbf{JY}_\mathbf{y}$

$$\mathbf{L}(\gamma) \simeq \mathbf{L}(\gamma_0) + \frac{\partial^R \mathbf{L}}{\partial \mathbf{v}^A}(\gamma_0) \wedge (\mathbf{v}^A - \mathbf{v}^A),$$

where $\gamma = (x^\mu, \mathbf{y}^A, \mathbf{w}^A)$ and $\gamma_0 = (x^\mu, \mathbf{y}^A, \mathbf{v}^A)$, defines an element of the d-jet dual \mathbf{JY}^*

$$-\mathbf{L}(\gamma) \simeq -\mathbf{L} - \frac{\partial^R \mathbf{L}}{\partial \mathbf{v}^A} \wedge (\mathbf{w}^A - \mathbf{v}^A) = (-1)^{n-q} \mathbf{p}_A \wedge \mathbf{w}^A + \mathbf{h}.$$

So, the Legendre transformation \mathbb{FL} is a map from \mathbf{JY} to its d-jet dual \mathbf{JY}^* given by

$$\mathbb{FL} : (x^\mu, \mathbf{y}^A, \mathbf{v}^A) \mapsto \left(x^\mu, \mathbf{y}^A, \mathbf{p}_A = (-1)^{n-q-1} \frac{\partial^R \mathbf{L}}{\partial \mathbf{v}^A}, \mathbf{h} = \frac{\partial^R \mathbf{L}}{\partial \mathbf{v}^A} \wedge \mathbf{v}^A - \mathbf{L} \right). \quad (6)$$

Let $\mathbf{P} \subset \mathbf{JY}^*$ be a submanifold of the d-jet dual defined as an image of the Legendre transformation, i.e. $\mathbf{P} := \mathbb{FL}(\mathbf{JY})$, and let $\mathbb{FL}_{\mathbf{P}}$ be \mathbb{FL} considered as a map onto \mathbf{P} . If $\mathbb{FL}_{\mathbf{P}} : \mathbf{JY} \rightarrow \mathbf{P}$ is a diffeomorphism then the Lagrangian is called regular and then there also exists its inverse $\mathbb{FL}_{\mathbf{P}}^{-1} : \mathbf{P} \rightarrow \mathbf{JY}$ which allows to express the generalized velocities \mathbf{v}^A as functions of $x^\mu, \mathbf{y}^A, \mathbf{p}_A$. The fibre bundle $\mathfrak{p}_{\mathbf{Y}, \mathbf{P}} : \mathbf{P} \rightarrow \mathbf{Y}$ is called the dynamical phase bundle, or simply the phase bundle. The phase bundle \mathbf{P} is locally described by the coordinates $(x^\mu, \mathbf{y}^A, \mathbf{p}_A)$, where the coordinates \mathbf{p}_A are called the canonical momenta. From now until the section IV we are dealing with regular Lagrangians only.

There can be defined the canonical forms in regular case by using the canonical forms on \mathbf{JY}^* and the Legendre transformation. The (dynamical) Cartan-Poincaré n -form Θ on \mathbf{P} is

$$\Theta := \theta|_{\mathbf{P}} = (-1)^{n-q} \mathbf{p}_A \wedge d\mathbf{y}^A + \mathbf{H}, \quad (7)$$

where \mathbf{M} -horizontal map $\mathbf{H} : \mathbf{P} \rightarrow \Lambda^n \mathbf{P}$ is the Hamiltonian defined by

$$\mathbf{H} \circ \mathbb{FL} = \frac{\partial^R \mathbf{L}}{\partial \mathbf{v}^A} \wedge \mathbf{v}^A - \mathbf{L}. \quad (8)$$

The (dynamical) multisymplectic $(n+1)$ -form is given in a standard way as

$$\Omega := -d\Theta = (-1)^{n-q-1} d\mathbf{p}_A \wedge d\mathbf{y}^A - d\mathbf{H}. \quad (9)$$

The word 'multisymplectic' means following. Let \mathbf{P} be a manifold and let there be defined nondegenerate closed $(n+1)$ -form Ω on \mathbf{P} , then the pair (\mathbf{P}, Ω) is called the multisymplectic manifold. We assume that Ω is also exact, i.e. there exists the n -form Θ , such that $\Omega = -d\Theta$. It is obvious that if $n = 1$ (spacetime is only time), then the definition reduces to the usual definition of the symplectic manifold.

The main goal of this subsection is to derive the covariant Hamilton equations of motion from the variational action principle, find the relation between them and the symmetry group of the system. This relation rarely exists, so the sufficient conditions should be derived. It will be shown in the next section that in the case of the Einstein-Cartan theory the conditions are satisfied. Let $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ be the section of the phase bundle $\mathbf{P} \rightarrow \mathbf{M}$. Then the Lagrangian evaluated on the section φ is given by the pullback of the Cartan-Poincaré n -form Θ

$$\mathbf{L}(x^\mu, \mathbf{y}_\varphi^A, \mathbf{p}_A^\varphi, d\mathbf{y}_\varphi^A) = -\varphi^*(\Theta) = (-1)^{n-q-1} \mathbf{p}_A^\varphi \wedge d\mathbf{y}_\varphi^A - \mathbf{H}(x^\mu, \mathbf{y}_\varphi^A, \mathbf{p}_A^\varphi). \quad (10)$$

The action of the system in the state given by the section φ on the spacetime interval $\mathbf{M}_I = \cup_{t \in I} \Sigma_t$, where $I = \langle t_{ini}, t_{fin} \rangle$, is given by

$$S_I(\varphi) = \int_{\mathbf{M}_I} -\varphi^*(\Theta). \quad (11)$$

There exists a following theorem¹¹

Theorem II.2. Section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is a stationary point of the action integral (11) if and only if for all vector fields $\mathbf{v} \in \text{Sec}(\mathbf{P}, \mathbf{TP})$ on the phase bundle \mathbf{P} the following expression is satisfied

$$\varphi^*(i_{\mathbf{v}}\Omega) = 0. \quad (12)$$

Proof. A variation of the action integral (11) is, where $\varphi + \delta\varphi = (x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x}) + \delta\mathbf{y}^A(\mathbf{x}), \mathbf{p}_A^\varphi(\mathbf{x}) + \delta\mathbf{p}_A(\mathbf{x}))$ means small variation of the section φ ,

$$\delta S_I(\varphi) = \int_{\mathbf{M}_I} \left[\delta\mathbf{p}_A \wedge \varphi^* \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) + \varphi^* \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \wedge \delta\mathbf{y}^A \right].$$

The stationarity condition of (11) yields the covariant Hamilton equations

$$0 = \varphi^* \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) = (-1)^{n-q-1} d\mathbf{y}_\varphi^A(\mathbf{x}) - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A}(x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x}), \mathbf{p}_A^\varphi(\mathbf{x})), \quad (13)$$

$$0 = \varphi^* \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) = -d\mathbf{p}_A^\varphi(\mathbf{x}) - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A}(x^\mu(\mathbf{x}), \mathbf{y}_\varphi^A(\mathbf{x}), \mathbf{p}_A^\varphi(\mathbf{x})).$$

Let $\mathbf{v} \in \text{Sec}(\mathbf{P}, \mathbf{TP})$ be arbitrary vector field on \mathbf{P} . It can be expressed as

$$\mathbf{v} = \xi^\mu \partial_\mu + v_{(\nu)q}^A \partial_{y_{(\nu)q}^A} + w_{A(\nu)q+1} \partial_{p_{A(\nu)q+1}} \equiv \xi + \mathbf{v}^A \partial_{\mathbf{y}^A} + \mathbf{w}_A \partial_{\mathbf{p}_A}$$

Action of the interior product of \mathbf{v} with the dynamical multisymplectic form Ω is given by

$$\begin{aligned} i_{\mathbf{v}}\Omega &= \mathbf{w}_A \wedge \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) - \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \wedge \mathbf{v}^A - \\ &\quad - i_\xi \left\{ \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \wedge \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) \right\} + \\ &\quad + i_\xi \left(\frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \wedge \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) \end{aligned} \quad (14)$$

The last term is vanishing since the object in the bracket is horizontal $(n+1)$ -form over \mathbf{M} . The rest is obvious. If the pullback of $i_{\mathbf{v}}\Omega$ along $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is vanishing for arbitrary \mathbf{v} then

$$\varphi^*(i_{\mathbf{v}}\Omega) = 0 \Rightarrow \varphi^* \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) = 0 \text{ and } \varphi^* \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) = 0$$

which means the section φ must satisfy the Hamilton equations (13). And vice versa if the equations (13) are satisfied then $\varphi^*(i_{\mathbf{v}}\Omega) = 0$ for arbitrary $\mathbf{v} \in \text{Sec}(\mathbf{P}, \mathbf{TP})$. \square

Let G be a certain subgroup of the group of all d -jet diffeomorphisms $\mathbf{d}\text{-Diff}(\mathbf{P})$ on the phase bundle $\mathbf{P} \rightarrow \mathbf{M}$ and let for all $\eta_{\mathbf{P}} \in G$ there exists generating vector field $\mathbf{v} \in \text{Sec}(\mathbf{P}, \mathbf{TP})$. Linear span of such vector fields forms Lie algebra $\text{alg}(G) \subset \text{Sec}(\mathbf{P}, \mathbf{TP})$ of the group G . The group is called symmetry of the physical system if every element $\eta_{\mathbf{P}} \in G$ keeps the dynamical Cartan-Poincaré n -form Θ invariant

$$(\eta_{\mathbf{P}}^{-1})^* \Theta = \Theta \quad (15)$$

For arbitrary vector $\mathbf{v} \in \text{alg}(G)$ this condition yields

$$\mathbb{L}_{\mathbf{v}}\Theta = 0. \quad (16)$$

Let $Q \in \text{Sec}(\mathbf{P}, \Lambda^{n-1}\mathbf{P})$ be a $(n-1)$ -form on \mathbf{P} and let there exists a vector field $\mathbf{v} \in \text{Sec}(\mathbf{P}, \mathbf{TP})$ such that

$$i_{\mathbf{v}}\Omega = dQ,$$

then \mathbf{v} is the vector field associated to the $(n-1)$ -form Q . Since for $\forall \varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$

$$\varphi^*(i_{\mathbf{v}}\Omega) = d\varphi^*Q,$$

the theorem II.2 implies that if φ is the solution of the Hamilton equations (13) then $d\varphi^*Q = 0$, i.e. φ^*Q is a current of conserved quantity. This yields the famous Noether's theorem.

Theorem II.3. *Let G be the symmetry group of the physical system and let φ be the solution of the Hamilton equations (13) then for arbitrary $\mathbf{v} \in \mathfrak{alg}(G)$ there exists current $\mathcal{J}_{\mathbf{v}} = (i_{\mathbf{v}}\Theta)$, called Noether's current, such that the equation $d\varphi^*\mathcal{J}_{\mathbf{v}} = 0$ is satisfied.*

Proof. Condition (16) implies

$$0 = \mathbb{L}_{\mathbf{v}}\Theta = i_{\mathbf{v}}d\Theta + d i_{\mathbf{v}}\Theta \quad \Rightarrow \quad -i_{\mathbf{v}}d\Theta = i_{\mathbf{v}}\Omega = d i_{\mathbf{v}}\Theta$$

This with the comments above proves the theorem. \square

Now, turn our attention to the converse of this theorem. In the case when the symmetry group G is sufficiently large, this of course depends on the considered physical system, the answer is in affirmative. Let $\mathfrak{alg}_{\mathbf{p}}(G) = \text{Span}\{\mathbf{v}|_{\mathbf{p}} \in T_{\mathbf{p}}\mathbf{P}; \mathbf{v} \in \mathfrak{alg}(G)\}$ denote a vector space spanned on vectors of the Lie algebra $\mathfrak{alg}(G)$ settled in the point $\mathbf{p} \in \mathbf{P}$. Group $G \subset \mathbf{d}\text{-}\mathfrak{Diff}(\mathbf{P})$ called vertically transitive¹¹ if $V_{\mathbf{p}}\mathbf{P} \subset \mathfrak{alg}_{\mathbf{p}}(G)$ for $\forall \mathbf{p} \in \mathbf{P}$.

Theorem II.4. *Let G be vertically transitive symmetry group of the system and let for $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ the condition*

$$\forall \mathbf{v} \in \mathfrak{alg}(G) : d\varphi^*\mathcal{J}_{\mathbf{v}} = 0$$

be satisfied, then φ is a solution of the Hamilton equations (13).

Proof. Since $V_{\mathbf{p}}\mathbf{P} \subset \mathfrak{alg}_{\mathbf{p}}(G)$ for $\forall \mathbf{p} \in \mathbf{P}$ then for all $\mathbf{v} \in \mathfrak{alg}(G)$ such that $\mathbf{v}|_{\mathbf{p}} = \mathbf{v}^A \partial_{\mathbf{y}^A} + \mathbf{w}_A \partial_{\mathbf{p}_A} \in V_{\mathbf{p}}\mathbf{P}$ one gets

$$d\mathcal{J}_{\mathbf{v}}|_{\mathbf{p}} = i_{\mathbf{v}}\Omega|_{\mathbf{p}} = \left(\mathbf{w}_A \wedge \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) - \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \wedge \mathbf{v}^A \right) \Big|_{\mathbf{p}}.$$

The pullback along the section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ gives

$$\begin{aligned} (d\varphi^*\mathcal{J}_{\mathbf{v}})|_{\mathbf{x}} &= \varphi^* \left(\mathbf{w}_A \wedge \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) - \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \wedge \mathbf{v}^A \right) \Big|_{\mathbf{x}} \\ &= (\varphi^*\mathbf{w}_A)|_{\mathbf{x}} \wedge \left(\varphi^* \left((-1)^{n-q-1} d\mathbf{y}^A - \frac{\partial^L \mathbf{H}}{\partial \mathbf{p}_A} \right) \right) \Big|_{\mathbf{x}} - \left(\varphi^* \left(-d\mathbf{p}_A - \frac{\partial^R \mathbf{H}}{\partial \mathbf{y}^A} \right) \right) \Big|_{\mathbf{x}} \wedge (\varphi^*\mathbf{v}_A)|_{\mathbf{x}} \end{aligned}$$

Now, it is easy to see, that the arbitrariness of $\mathbf{v}|_{\mathbf{p}} \in V_{\mathbf{p}}(\mathbf{P})$ and the condition $(d\varphi^*\mathcal{J}_{\mathbf{v}})|_{\mathbf{x}} = 0$ for all $\mathbf{x} \in \mathbf{M}$ imply that φ is the solution of the Hamilton equations (13). \square

Group $G \subset \mathbf{d}\text{-}\mathfrak{Diff}(\mathbf{P})$ is called localizable if for every pair (Σ_1, Σ_2) of disjoint embeddings of Σ and every vector $\mathbf{v} \in \mathfrak{alg}(G)$ there exists vector $\mathbf{w} \in \mathfrak{alg}(G)$ such that $\mathbf{v}|_{\Sigma_1} = \mathbf{w}|_{\Sigma_1}$ and $\mathbf{w}|_{\Sigma_2} = 0$. Localizable symmetry group is called gauge group. With these in hands we can formulate the second Noether's theorem of integral formulation of the equations of the motion.

Theorem II.5. *Let G be the vertically transitive gauge group G of the physical system. Section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is the solution of the Hamilton equations of motion (13) if and only if for $\forall \mathbf{v} \in \text{alg}(G)$ and all embeddings Σ_0 of Σ into \mathbf{M} the integral*

$$\int_{\Sigma_0} \varphi^* \mathcal{J}_{\mathbf{v}} = 0$$

is vanishing.

Proof. If the integral $\int_{\Sigma_0} \varphi^* \mathcal{J}_{\mathbf{v}} = 0$ is vanishing for all embeddings Σ_0 and all currents $\mathcal{J}_{\mathbf{v}}$ then we have for arbitrary foliation $\mathbf{M}_{\mathbf{I}} \simeq \Sigma \times \mathbf{I}$, where $\mathbf{I} = \langle t_{\text{ini}}, t_{\text{fin}} \rangle$, an identity

$$\begin{aligned} 0 &= \int_{\Sigma_{\text{fin}}} \varphi^* \mathcal{J}_{\mathbf{v}} - \int_{\Sigma_{\text{ini}}} \varphi^* \mathcal{J}_{\mathbf{v}} = \oint_{\partial \mathbf{M}_{\mathbf{I}}} \varphi^* \mathcal{J}_{\mathbf{v}} \\ &= \int_{\mathbf{M}_{\mathbf{I}}} \mathbf{d}(\varphi^* \mathcal{J}_{\mathbf{v}}). \end{aligned}$$

The arbitrariness of the foliation $\mathbf{M}_{\mathbf{I}}$ implies for all $\mathcal{J}_{\mathbf{v}}$

$$0 = \mathbf{d}(\varphi^* \mathcal{J}_{\mathbf{v}})$$

and due to theorem II.4 the section φ is the solution of the Hamilton equations of motion.

Conversely if the section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is the solution of the Hamilton equations then $0 = \mathbf{d}(\varphi^* \mathcal{J}_{\mathbf{v}})$ for all $\mathcal{J}_{\mathbf{v}}$ and therefore

$$0 = \int_{\Sigma_{\text{fin}}} \varphi^* \mathcal{J}_{\mathbf{v}} - \int_{\Sigma_{\text{ini}}} \varphi^* \mathcal{J}_{\mathbf{v}}.$$

Since G is localizable we may choose \mathbf{v} being vanishing on Σ_{ini} and hence

$$0 = \int_{\Sigma_{\text{fin}}} \varphi^* \mathcal{J}_{\mathbf{v}}$$

and arbitrariness of Σ_{fin} yields the result. \square

NOTE: The compactness of Σ plays crucial role in the theorem, if Σ is noncompact then one must take into account boundary terms, e.g. for asymptotically flat spacetimes these yield in general nonvanishing global charges (Energy-momentum "tensor" at infinity, total electric charge, ...).

III. LOCAL COVARIANT POISSON BRACKETS

In the canonical quantization Poisson bracket plays a crucial role. Indeed, a certain set of basic kinematical variables equipped with Poisson bracket forms an algebra, called fundamental, and the first step of any approach to the quantization is to find its representation on an appropriate Hilbert space. Therefore it is important to introduce Poisson bracket also in the field theory. As we will see we can define two types of brackets in the field theory ($n \geq 2$). The first one, called local covariant Poisson bracket, is defined in every point of the phase space. Representation of the local fundamental algebra can be viewed as the first quantization in the context of the covariant

quantization, but usually this step is called pre-quantization^{15,16}. The task of the section is to define local covariant Poisson bracket among covariant observables and also explore some of their algebraic properties.

In the previous section we have introduced two basic multisymplectic manifolds associated with the given physical system. The first one given by the d-jet dual \mathbf{JY}^* and its canonical multisymplectic form ω defined in (5) was playing role in the kinematical description. The second one given by the image of the Legendre transformation $\mathbf{P} = \mathbb{FL}(\mathbf{JY})$ equipped with the multisymplectic form Ω defined in (9) was used for the formulation of the Hamilton equations of motion (13). For a while we are not going to distinguish between them and explore some of their algebraic properties simultaneously. Let $\mathfrak{p}_{\mathbf{M}, \mathbf{F}} : \mathbf{F} \rightarrow \mathbf{M}$ be given fibre bundle over the spacetime \mathbf{M} , e.g. $\mathbf{F} = \mathbf{JY}^*$ or $\mathbf{F} = \mathbf{P}$, and let there be given exact $(n + 1)$ -multisymplectic form $\omega = -\mathbf{d}\theta$ over it, e.g. $\omega = \omega$ for \mathbf{JY}^* or $\omega = \Omega$ for \mathbf{P} . In standard symplectic case to every hamiltonian observable is there associated the hamiltonian vector field and it is similar in the multisymplectic case. We say that $\mathbf{v} \in \text{Sec}(\mathbf{F}, \mathbf{TF})$ is hamiltonian vector field if there exists hamiltonian $(n - 1)$ -form, or observable, $\mathbf{Q} \in \text{Sec}(\mathbf{F}, \Lambda^{n-1}\mathbf{F})$ such that

$$i_{\mathbf{v}}\omega = \mathbf{d}\mathbf{Q}. \quad (17)$$

Set of all hamiltonian vector fields or set of all hamiltonian $(n - 1)$ -forms is denoted by $\text{Ham}_{\omega}^1\mathbf{TF}$ or $\text{Ham}_{\omega}^{n-1}\mathbf{F}$, respectively.

The local covariant Poisson bracket for observables $\mathbf{Q}_1, \mathbf{Q}_2 \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$ is defined by¹³

$$\{\mathbf{Q}_1, \mathbf{Q}_2\} = i_{\mathbf{v}_1}i_{\mathbf{v}_2}\omega + \mathbf{d}(i_{\mathbf{v}_1}\mathbf{Q}_2 - i_{\mathbf{v}_2}\mathbf{Q}_1 - i_{\mathbf{v}_1}i_{\mathbf{v}_2}\theta), \quad (18)$$

where $\mathbf{v}_1, \mathbf{v}_2 \in \text{Ham}_{\omega}^1\mathbf{TF}$ are hamiltonian vectors related to $\mathbf{Q}_1, \mathbf{Q}_2$, respectively. The local covariant Poisson bracket defined in¹³ is equal up to the sign to local covariant Poisson bracket defined here and bracket in¹³ is also defined for, so-called, Poisson forms with degrees less then $n - 1$, but it is sufficient for our purposes to consider only (18). Local Poisson bracket defined by (18) satisfied Jacobi identity,¹³ theorem 3.8,

$$\{\mathbf{Q}_1, \{\mathbf{Q}_2, \mathbf{Q}_3\}\} + \{\mathbf{Q}_3, \{\mathbf{Q}_1, \mathbf{Q}_2\}\} + \{\mathbf{Q}_2, \{\mathbf{Q}_3, \mathbf{Q}_1\}\} = 0,$$

where $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$. We should note here that $(\text{Ham}_{\omega}^{n-1}\mathbf{F}, \{\cdot, \cdot\})$ is Lie algebra only, since there is no multiplication on $\text{Ham}_{\omega}^{n-1}\mathbf{F}$.

We have seen in the previous section that Noether's currents $\mathcal{J}_{\mathbf{v}}$ are generated by vector fields preserving Cartan-Poincaré form θ , i.e. if $\mathcal{L}_{\mathbf{v}}\theta = 0$ then for $i_{\mathbf{v}}\theta = \mathcal{J}_{\mathbf{v}}$ we have $i_{\mathbf{v}}\omega = \mathbf{d}\mathcal{J}_{\mathbf{v}}$. There exists Lie algebra defined on the set of all Noether's vector fields with product given by Lie bracket $\llbracket \cdot, \cdot \rrbracket$ on \mathbf{TF} . This algebra is naturally represented on the set of all Noether's currents.

Lemma III.1. *Let $\mathbf{v}_1, \mathbf{v}_2$ be Noether's vectors, i.e. satisfying $\mathcal{L}_{\mathbf{v}_1}\theta = \mathcal{L}_{\mathbf{v}_2}\theta = 0$, then (18) yields*

$$\{\mathcal{J}_{\mathbf{v}_1}, \mathcal{J}_{\mathbf{v}_2}\} = \mathcal{J}_{\llbracket \mathbf{v}_1, \mathbf{v}_2 \rrbracket}, \quad (19)$$

where $\llbracket \cdot, \cdot \rrbracket$ means Lie bracket on \mathbf{TF} .

Proof. Direct calculation yields

$$\begin{aligned} \{\mathcal{J}_{\mathbf{v}_1}, \mathcal{J}_{\mathbf{v}_2}\} &= -i_{\mathbf{v}_1}i_{\mathbf{v}_2}\mathbf{d}\theta - \mathbf{d}i_{\mathbf{v}_2}i_{\mathbf{v}_1}\theta \\ &= -i_{\mathbf{v}_2}\mathbf{d}i_{\mathbf{v}_1}\theta - \mathbf{d}i_{\mathbf{v}_2}i_{\mathbf{v}_1}\theta \\ &= -\mathcal{L}_{\mathbf{v}_2}i_{\mathbf{v}_1}\theta = i_{\llbracket \mathbf{v}_1, \mathbf{v}_2 \rrbracket}\theta = \mathcal{J}_{\llbracket \mathbf{v}_1, \mathbf{v}_2 \rrbracket}, \end{aligned}$$

where we used relation $i_{\llbracket \mathbf{v}_1, \mathbf{v}_2 \rrbracket} = \mathcal{L}_{\mathbf{v}_1}i_{\mathbf{v}_2} - i_{\mathbf{v}_2}\mathcal{L}_{\mathbf{v}_1}$. □

We have constructed two different multisymplectic structures for given physical system (\mathbf{Y}, \mathbf{L}) . Now, we are curious how these structures are related. Let $(\mathbf{F} = \mathbf{JY}^*, \omega)$ with related algebra of observables $(\text{Ham}_\omega^{n-1} \mathbf{F}, \llbracket \cdot, \cdot \rrbracket_\omega)$ or (\mathbf{P}, Ω) with $(\text{Ham}_\Omega^{n-1} \mathbf{P}, \llbracket \cdot, \cdot \rrbracket_\Omega)$ be kinematical or dynamical multisymplectic manifold, respectively. Since \mathbf{F} or \mathbf{P} can be locally coordinatized by $(x^\mu, \mathbf{y}^A, \mathbf{p}_A, \mathbf{h})$ or $(x^\mu, \mathbf{y}^A, \mathbf{p}_A)$, respectively, we have a bundle chain

$$\mathbf{F} \xrightarrow{\mathbb{P}_{\mathbf{P}, \mathbf{F}}} \mathbf{P} \xrightarrow{\mathbb{P}_{\mathbf{H}, \mathbf{P}}} \mathbf{M} \quad \text{and} \quad \mathbb{P}_{\mathbf{M}, \mathbf{F}} = \mathbb{P}_{\mathbf{M}, \mathbf{P}} \circ \mathbb{P}_{\mathbf{P}, \mathbf{F}}.$$

Phase space \mathbf{P} can be embedded into the kinematical multisymplectic space \mathbf{F} by Hamilton map $\mathbb{F}\mathbf{H} : \mathbf{P} \rightarrow \mathbf{F}$ defined by

$$\mathbb{F}\mathbf{H} : (\mathbf{x}, \mathbf{y}^A, \mathbf{p}_A) \mapsto (\mathbf{x}, \mathbf{y}^A, \mathbf{p}_A, \mathbf{h} = \mathbf{H}), \quad (20)$$

where \mathbf{H} is Hamiltonian defined by (8).

Let $\mathbf{v} \in \mathbf{TF}$ or $\mathbf{v} \in \mathbf{TP}$ then we say that \mathbf{v} is \mathbf{M} -vertical if $(\mathbb{P}_{\mathbf{M}, \mathbf{F}})_* \mathbf{v} = 0$ or $(\mathbb{P}_{\mathbf{M}, \mathbf{P}})_* \mathbf{v} = 0$ and we denote subbundle of such vectors by $\mathbf{M}\text{-VF} \subset \mathbf{TF}$ or $\mathbf{M}\text{-VP} \subset \mathbf{TP}$, respectively. Vector $\mathbf{v} \in \mathbf{TF}$ is called \mathbf{H} -vertical if $(\mathbb{P}_{\mathbf{P}, \mathbf{F}})_* \mathbf{v} = 0$ and subbundle of such vectors is denoted by $\mathbf{H}\text{-VF} \subset \mathbf{TF}$. $(\mathbb{P})_* \mathbf{v}$ means pushforward of the vector \mathbf{v} along map \mathbb{P} . Let $\alpha \in \Lambda \mathbf{F}$ or $\alpha \in \Lambda \mathbf{P}$ we say that α is \mathbf{M} -horizontal if α is annihilated by all \mathbf{M} -vertical vectors and subbundle of such forms is denoted by $\mathbf{M}\text{-horF} \subset \Lambda \mathbf{F}$ or $\mathbf{M}\text{-horP} \subset \Lambda \mathbf{P}$, respectively. Let $\alpha \in \Lambda \mathbf{F}$ we say that α is \mathbf{H} -horizontal if α is annihilated by all \mathbf{H} -vertical vectors and subbundle of such forms is denoted by $\mathbf{H}\text{-horF} \subset \Lambda \mathbf{F}$.

Let $\mathbf{Q} \in \text{Ham}_\omega^{n-1} \mathbf{F}$ be kinematical observable on \mathbf{F} and $\mathbf{v} \in \text{Ham}_\omega^1 \mathbf{TF}$ be its related hamiltonian vector. Since relation between bundles \mathbf{F} and \mathbf{P} is given by projection $\mathbb{P}_{\mathbf{P}, \mathbf{F}}$ and Hamilton map $\mathbb{F}\mathbf{H}$ it is important to explore how general \mathbf{Q} depends on \mathbf{h} . Let us denote $\mathbf{h} = \bar{h} \mathbf{d}\Sigma$. We can decompose \mathbf{Q} as

$$\mathbf{Q} = \alpha + \mathbf{d}\bar{h} \wedge \beta,$$

where $\alpha \in \text{Sec}(\mathbf{F}, \mathbf{H}\text{-hor}^{n-1} \mathbf{F})$ and $\beta \in \text{Sec}(\mathbf{F}, \mathbf{H}\text{-hor}^{n-2} \mathbf{F})$ are certain \mathbf{H} -horizontal form fields on \mathbf{F} . Since $\mathbf{d}\bar{h} \wedge \beta = \mathbf{d}(\bar{h}\beta) - \bar{h} \wedge \mathbf{d}\beta$ there exists $\mathbf{Q}' \in \text{Ham}_\omega^{n-1} \mathbf{F} \cap \text{Sec}(\mathbf{F}, \mathbf{H}\text{-hor}^{n-1} \mathbf{F})$ such that \mathbf{Q}' and \mathbf{v} are related by (17). Therefore we can consider each observable $\mathbf{Q} \in \text{Ham}_\omega^{n-1} \mathbf{F}$ being \mathbf{H} -horizontal without lack of generality. Let $\mathbf{P}_{\mathbb{F}\mathbf{H}} = \mathbb{F}\mathbf{H}(\mathbf{P})$ be image of the Hamilton map then \mathbf{Q} can be decomposed as

$$\mathbf{Q} = \mathbf{Q}_{\mathbf{P}} + \mathbf{Q}_0,$$

where $\mathbf{Q}_{\mathbf{P}} = (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(\mathbb{F}\mathbf{H})^* \mathbf{Q}$ being \mathbf{h} independent and $\mathbf{Q}_0|_{\mathbf{P}_{\mathbb{F}\mathbf{H}}} = 0$. Its associated vector field $\mathbf{v} \in \text{Ham}_\omega^1 \mathbf{TF}$ can be written as

$$\mathbf{v} = \xi^\mu \partial_\mu + w_{(\mu)_q}^A \partial_{y_{(\mu)_q}^A} + w_{A(\mu)^{q+1}} \partial_{p_{A(\mu)^{q+1}}} + \bar{u} \partial_{\bar{h}} \quad (21)$$

$$= \xi + \mathbf{w} + \mathbf{u} \partial_{\mathbf{h}} \quad (22)$$

and from (17) we get

$$i_\xi (-1)^{n-q-1} \mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A + \mathbf{d}\bar{h} \wedge i_\xi \mathbf{d}\Sigma + i_{\mathbf{w}} (-1)^{n-q-1} \mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A - \mathbf{u} = \mathbf{d}^{\mathbf{P}} \mathbf{Q}_{\mathbf{P}} + \mathbf{d}^{\mathbf{P}} \mathbf{Q}_0 + \mathbf{d}\bar{h} \wedge \frac{\partial \mathbf{Q}_0}{\partial \bar{h}}, \quad (23)$$

where we used the notation

$$\mathbf{d}^{\mathbf{P}} \alpha = \mathbf{d}x^\mu \wedge \alpha_{,\mu} + \mathbf{d}y_{(\mu)_q}^A \wedge \frac{\partial \alpha}{\partial y_{(\mu)_q}^A} + \mathbf{d}p_{A(\mu)^{q+1}} \wedge \frac{\partial \alpha}{\partial p_{A(\mu)^{q+1}}}$$

for short. If we compare terms in (23) containing $\mathbf{d}\bar{h}$ we get

$$i_{\xi}\mathbf{d}\Sigma = \frac{\partial \mathbf{Q}_0}{\partial \bar{h}}.$$

The rest of terms yields that we have a following chain of projectable vectors $\mathbf{v} \rightarrow \xi + \mathbf{w} \rightarrow \xi$ on $\mathbf{F} \rightarrow \mathbf{P} \rightarrow \mathbf{M}$ and hence $\mathbf{Q}_0 = i_{\xi}(\mathbf{h} - \mathbf{H})$ taking into account the condition $\mathbf{Q}_0|_{\mathbb{P}_{\mathbf{FH}}} = 0$.

Lemma III.2. *Let $\mathbf{Q}_1^{\mathbf{F}}, \mathbf{Q}_2^{\mathbf{F}} \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$ and $(\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(\mathbb{F}\mathbf{H})^*\mathbf{Q}_1^{\mathbf{F}} = (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(\mathbb{F}\mathbf{H})^*\mathbf{Q}_2^{\mathbf{F}}$ then $\mathbf{Q}_1^{\mathbf{F}} = \mathbf{Q}_2^{\mathbf{F}}$.*

Proof. Let $\mathbf{v}_i = \xi_i + \mathbf{w}_i + \mathbf{u}_i\partial_{\mathbf{h}}$ be the decomposition, similar to (22), of the hamiltonian vector $\mathbf{v}_i \in \text{Ham}_{\omega}^1\mathbf{T}\mathbf{F}$ associated to the observable $\mathbf{Q}_i^{\mathbf{F}} \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$, where $i = 1, 2$. Each $\mathbf{Q}_i^{\mathbf{F}}$ can be written as

$$\mathbf{Q}_i^{\mathbf{F}} = \mathbf{Q}_{\mathbf{P}} + \mathbf{Q}_i, \text{ where } \mathbf{Q}_{\mathbf{P}} = (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(\mathbb{F}\mathbf{H})^*\mathbf{Q}_i^{\mathbf{F}}.$$

For $\mathbf{Q}_i = i_{\xi_i}(\mathbf{h} - \mathbf{H})$ we get

$$\mathbf{dQ}_i^{\mathbf{F}} = \mathbf{dQ}_{\mathbf{P}} + i_{\mathbf{d}\xi_i}(\mathbf{h} - \mathbf{H}) - i_{\xi_i}\mathbf{d}(\mathbf{h} - \mathbf{H}) \quad (24)$$

and also

$$i_{\mathbf{v}_i}\omega = -i_{\xi_i}\mathbf{d}\mathbf{h} + i_{\xi_i}(-1)^{n-q-1}\mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A + i_{\mathbf{w}_i}(-1)^{n-q-1}\mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A - \mathbf{u}_i. \quad (25)$$

Since the second term $i_{\xi_i}(-1)^{n-q-1}\mathbf{d}\mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A$ in (25) can be canceled only by part of $\mathbf{dQ}_{\mathbf{P}}$ in (24) we have that $\xi_1 = \xi_2$ and therefore $\mathbf{Q}_1 = \mathbf{Q}_2$. \square

This lemma shows that there exists at most one observable $\mathbf{Q}^{\mathbf{F}} \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$ for given form $\mathbf{Q}^{\mathbf{P}} \in \text{Sec}(\mathbf{P}, \Lambda^{n-1}\mathbf{P})$ over \mathbf{P} and vice versa $\text{Ham}_{\omega}^{n-1}\mathbf{F}$ defines subspace of kinematical observables $\text{Ham}_{\omega}^{n-1}\mathbf{P} = (\mathbb{F}\mathbf{H})^*\text{Ham}_{\omega}^{n-1}\mathbf{F}$ in the set of all $(n-1)$ -forms $\text{Sec}(\mathbf{P}, \Lambda^{n-1}\mathbf{P})$ on dynamical phase space \mathbf{P} . Let $\mathbf{Q}_i^{\mathbf{P}} \in \text{Ham}_{\omega}^{n-1}\mathbf{P}$ and let $\mathbf{Q}_i^{\mathbf{F}}$ be its kinematical extension on \mathbf{F} then we can define the local kinematical Poisson bracket on $\text{Ham}_{\omega}^{n-1}\mathbf{P}$ by

$$\llbracket \mathbf{Q}_1^{\mathbf{P}}, \mathbf{Q}_2^{\mathbf{P}} \rrbracket_{\omega}^{\mathbf{P}} = (\mathbb{F}\mathbf{H})^*\llbracket \mathbf{Q}_1^{\mathbf{F}}, \mathbf{Q}_2^{\mathbf{F}} \rrbracket_{\omega}. \quad (26)$$

and get Lie algebra of kinematical observables $(\text{Ham}_{\omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$ on the dynamical phase space \mathbf{P} .

It seems that we have two local Poisson brackets on dynamical phase space \mathbf{P} : Kinematical algebra $(\text{Ham}_{\omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$ just defined and dynamical algebra $(\text{Ham}_{\Omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\Omega})$. Problem is that the dynamical bracket depends on the Hamiltonian and since the dynamical observables are constants of motion we can not measure any local information depending on time evolution by them. Following lemma shows that the dynamical algebra $(\text{Ham}_{\Omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\Omega})$ is subalgebra in $(\text{Ham}_{\omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$ therefore the bracket defined by (26) extends the algebra of observables $(\text{Ham}_{\Omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\Omega})$ on the dynamical phase space \mathbf{P} .

Lemma III.3. *$(\text{Ham}_{\Omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\Omega})$ is subalgebra in $(\text{Ham}_{\omega}^{n-1}\mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$.*

Proof. Let $\mathbf{Q}^{\mathbf{P}} \in \text{Ham}_{\Omega}^{n-1}\mathbf{P}$ and $\mathbf{V} \in \text{Ham}^1\mathbf{TP}$ be its associated hamiltonian vector. Goal is to find $\mathbf{Q}^{\mathbf{F}} \in \text{Ham}_{\omega}^{n-1}\mathbf{F}$ with associated vector $\mathbf{v} = \xi + \mathbf{w} + \mathbf{u}\partial_{\mathbf{h}} \in \text{Ham}_{\omega}^1\mathbf{F}$ such that $(\mathbb{F}\mathbf{H})^*\mathbf{Q}^{\mathbf{F}} = \mathbf{Q}^{\mathbf{P}}$. We already know that if $\mathbf{Q}^{\mathbf{F}}$ exists then it looks like $\mathbf{Q}^{\mathbf{F}} = (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*\mathbf{Q}^{\mathbf{P}} + i_{\xi}(\mathbf{h} - \mathbf{H})$. We have

$$\begin{aligned} i_{\mathbf{v}}\omega &= \mathbf{dQ}^{\mathbf{F}}, \\ i_{\mathbf{v}}((\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*\Omega - \mathbf{d}(\mathbf{h} - \mathbf{H})) &= (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(i_{\mathbf{V}}\Omega) + i_{\mathbf{d}\xi}(\mathbf{h} - \mathbf{H}) - i_{\xi}\mathbf{d}(\mathbf{h} - \mathbf{H}), \\ i_{\mathbf{v}}((\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*\Omega) - \mathbf{u} &= (\mathbb{P}_{\mathbf{P}, \mathbf{F}})^*(i_{\mathbf{V}}\Omega) + i_{\mathbf{d}\xi}(\mathbf{h} - \mathbf{H}). \end{aligned}$$

Since \mathbf{v} is \mathbf{H} -projectable the last equation yields $(\mathbb{p}_{\mathbf{P}, \mathbf{F}})_* \mathbf{v} = \mathbf{V}$ and $\mathbf{u} = -i_{\mathbf{d}\xi}(\mathbf{h} - \mathbf{H})$. Therefore $\mathbf{Q}^{\mathbf{F}} = (\mathbb{p}_{\mathbf{P}, \mathbf{F}})^* \mathbf{Q}^{\mathbf{P}} + i_{\xi}(\mathbf{h} - \mathbf{H})$ is wanted extension for all $\forall \mathbf{Q}^{\mathbf{P}} \in \text{Ham}_{\Omega}^{n-1} \mathbf{P}$. We have

$$\llbracket \mathbf{Q}_1^{\mathbf{F}}, \mathbf{Q}_2^{\mathbf{F}} \rrbracket_{\omega} = (\mathbb{p}_{\mathbf{P}, \mathbf{F}})^* \llbracket \mathbf{Q}_1^{\mathbf{P}}, \mathbf{Q}_2^{\mathbf{P}} \rrbracket_{\Omega} + i_{\llbracket \xi_1, \xi_2 \rrbracket}(\mathbf{h} - \mathbf{H})$$

for $\forall \mathbf{Q}_i^{\mathbf{P}} \in \text{Ham}_{\Omega}^{n-1} \mathbf{P}$ hence definition (26) yields directly $\llbracket \mathbf{Q}_1^{\mathbf{P}}, \mathbf{Q}_2^{\mathbf{P}} \rrbracket_{\omega}^{\mathbf{P}} = \llbracket \mathbf{Q}_1^{\mathbf{P}}, \mathbf{Q}_2^{\mathbf{P}} \rrbracket_{\Omega}$. \square

To simplify notation we write $(\text{Ham}^{n-1} \mathbf{F}, \llbracket \cdot, \cdot \rrbracket)$ or $(\text{Ham}^{n-1} \mathbf{P}, \llbracket \cdot, \cdot \rrbracket)$ instead of $(\text{Ham}_{\omega}^{n-1} \mathbf{F}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$ or $(\text{Ham}_{\omega}^{n-1} \mathbf{P}, \llbracket \cdot, \cdot \rrbracket_{\omega}^{\mathbf{P}})$, respectively. Let us construct some observables.

Lemma III.4. *Following forms are observables belonging to $\text{Ham}^{n-1} \mathbf{F}$ for $\forall \pi_A \in \text{Sec}(\mathbf{M}, \Lambda^{n-q-1} \mathbf{M})$, $\forall \Xi^A \in \text{sec}(\mathbf{M}, \Lambda^q \mathbf{M})$*

$$i) \quad \mathbf{y}(\pi) = \pi_A \wedge \mathbf{y}^A,$$

$$ii) \quad \mathbf{p}(\Xi) = \mathbf{p}_A \wedge \Xi^A,$$

where we write $\pi_A = \mathbb{p}_{\mathbf{M}, \mathbf{F}}^* \pi_A$ and similar for Ξ^A . The one only non-trivial local Poisson bracket is

$$\llbracket \pi_A \wedge \mathbf{y}^A, \mathbf{p}_B \wedge \Xi^B \rrbracket = \pi_A \wedge \Xi^A.$$

Proof. Direct calculation shows that vectors

$$\mathbf{v}(\pi) = \pi_A \partial_{\mathbf{p}_A} - (\mathbf{d}\pi_A \wedge \mathbf{y}^A) \partial_{\mathbf{h}}$$

and

$$\mathbf{v}(\Xi) = -\xi^A \partial_{\mathbf{y}^A} - (-1)^{n-q-1} (\mathbf{p}_A \wedge \Xi^A) \partial_{\mathbf{h}}$$

solve equation (17) for appropriate observables $\mathbf{y}(\pi)$ and $\mathbf{p}(\Xi)$, respectively. Since both observables are \mathbf{M} -horizontal and their hamiltonian vectors are vertical on $\mathbf{F} \rightarrow \mathbf{M}$ we have from (18) immediately

$$\llbracket \mathbf{p}(\Xi), \mathbf{p}(\Xi') \rrbracket = \llbracket \mathbf{y}(\pi), \mathbf{y}(\pi') \rrbracket = 0$$

and

$$\llbracket \mathbf{y}(\pi), \mathbf{p}(\Xi) \rrbracket = i_{\mathbf{v}(\pi)} \mathbf{d}(\mathbf{p}_A \wedge \Xi^A) = \pi_A \wedge \Xi^A.$$

\square

Lemma III.5. *There exist representation of $\mathcal{D}\text{iff}(\mathbf{M})$ on $\text{Ham}^{n-1} \mathbf{F}$ generated by Noether's currents $\mathcal{J}^{\mathbf{F}}(\xi)$. $\mathcal{J}^{\mathbf{F}}(\xi)$ are defined for every $\xi \in \text{alg}(\mathcal{D}\text{iff}(\mathbf{M})) \equiv \text{Sec}(\mathbf{M}, \mathbf{T}\mathbf{M})$ by*

$$\mathcal{J}^{\mathbf{F}}(\xi) = (-1)^{n-q} i_{\xi} \mathbf{p}_A \wedge \mathbf{d}\mathbf{y}^A + \mathbf{p}_A \wedge \mathbf{d}i_{\xi} \mathbf{y}^A + i_{\xi} \mathbf{h}$$

and satisfy identities

$$\llbracket \mathcal{J}^{\mathbf{F}}(\xi), \mathcal{J}^{\mathbf{F}}(\xi') \rrbracket = \mathcal{J}^{\mathbf{F}}(\llbracket \xi, \xi' \rrbracket), \quad (27)$$

$$\llbracket \mathbf{y}(\pi), \mathcal{J}^{\mathbf{F}}(\xi) \rrbracket = \mathbf{y}(-\mathcal{L}_{\xi} \pi), \quad (28)$$

$$\llbracket \mathbf{p}(\Xi), \mathcal{J}^{\mathbf{F}}(\xi) \rrbracket = \mathbf{p}(-\mathcal{L}_{\xi} \Xi). \quad (29)$$

Proof. At first we introduce following notation. If we denote

$$\mathbf{d}\xi = \xi_{,\nu}^{\mu} \mathbf{d}x^{\nu} \otimes \partial_{\mu},$$

then we can consider $\mathbf{d}\xi$ as object, not tensor, of the set of vector valued forms¹⁸, where the interior product of vector valued r -form $\beta = \sum_a \alpha_a \otimes \mathbf{v}_a \in \Lambda_{\mathbf{x}}^r \mathbf{X} \otimes \mathbf{T}_{\mathbf{x}} \mathbf{X}$ over $\mathbf{x} \in \mathbf{X}$, where \mathbf{X} is general manifold,

and $\forall a : \alpha_a \in \Lambda_{\mathbf{x}}^r \mathbf{X}, \mathbf{v}_a \in \mathbf{T}_{\mathbf{x}} \mathbf{X}$ is defined for $\forall \theta \in \Lambda_{\mathbf{x}} \mathbf{X}$ by

$$i_{\beta} \theta = \sum_a \alpha_a \wedge i_{\mathbf{v}_a} \theta.$$

Let $\eta \in \mathfrak{Diff}(\mathbf{M})$ then the map defined on kinematical bundle \mathbf{F} by

$$\eta_{\mathbf{F}} : (\mathbf{x}, \mathbf{y}^A, \mathbf{p}_A, \mathbf{h}) \mapsto (\eta(\mathbf{x}), (\eta^{-1}(\mathbf{x}))^* \mathbf{y}^A, (\eta^{-1}(\mathbf{x}))^* \mathbf{p}_A, (\eta^{-1}(\mathbf{x}))^* \mathbf{h})$$

can be extended into the d-jet diffeomorphism on \mathbf{F} due to theorem II.1. Let $\eta(\lambda) \in \mathfrak{Diff}(\mathbf{M})$ be one-parameter group generated by vector field $\xi \in \mathfrak{alg}(\mathfrak{Diff}(\mathbf{M}))$ then its d-jet extension $\eta_{\mathbf{F}}$ is generated by vector

$$\mathbf{w}(\xi) = \xi - i_{\mathbf{d}\xi} \mathbf{y}^A \partial_{\mathbf{y}^A} - i_{\mathbf{d}\xi} \mathbf{p}_A \partial_{\mathbf{p}_A} - i_{\mathbf{d}\xi} \mathbf{h} \partial_{\mathbf{h}}.$$

Since $\mathfrak{L}_{\mathbf{w}(\xi)} \theta = 0$ hence $\mathcal{J}^{\mathbf{F}}(\xi) = i_{\mathbf{w}(\xi)} \theta$ is kinematical Noether's charge. We have $[[\mathbf{w}(\xi), \mathbf{w}(\xi')]] = \mathbf{w}([[\xi, \xi']])$ therefore lemma III.1 yields (27). If we use that $\mathcal{J}^{\mathbf{F}}(\xi)$ is Noether's charge then the bracket (18) can be reduced into the form

$$[[\mathbf{Q}, \mathcal{J}^{\mathbf{F}}(\xi)]] = -\mathfrak{L}_{\mathbf{w}(\xi)} \mathbf{Q}$$

and relations $\mathfrak{L}_{\mathbf{w}(\xi)} \mathbf{y}^A = \mathfrak{L}_{\mathbf{w}(\xi)} \mathbf{p}_A = 0$ imply (28) and (29) immediately. \square

Theorem III.1. *General observable \mathbf{f} on \mathbf{F} can be locally decomposed as*

$$\mathbf{f} = \mathcal{J}^{\mathbf{F}}(\xi) + \mathbf{Q} + \mathbf{d}\alpha, \quad (30)$$

where $\mathcal{J}^{\mathbf{F}}(\xi)$ is given by previous lemma III.5, \mathbf{Q} is \mathbf{M} -horizontal simply differentiable $(n-1)$ -form depending only on $\mathbf{x}, \mathbf{y}^A, \mathbf{p}_A$ and α is arbitrary $(n-2)$ -form on \mathbf{F} . If each fibre $\mathbf{F}_{\mathbf{x}} = \mathbb{P}_{\mathbf{M}, \mathbf{F}}^{-1}(\mathbf{x})$ is contractible then (30) holds globally.

Proof. Let \mathbf{u} be general hamiltonian vector field associated to observable \mathbf{f} and let \mathbf{u} be given by $\mathbf{u} = \xi + \mathbf{u}^A \partial_{\mathbf{y}^A} + \mathbf{u}_A \partial_{\mathbf{p}_A} + \mu \partial_{\mathbf{h}}$. If we define vertical hamiltonian vector field

$$\mathbf{v} = \mathbf{u} - \mathbf{w}(\xi) = \mathbf{v}^A \partial_{\mathbf{y}^A} + \mathbf{v}_A \partial_{\mathbf{p}_A} + \nu \partial_{\mathbf{h}},$$

then we can decompose $\mathbf{f} = \mathcal{J}^{\mathbf{F}}(\xi) + \tilde{\mathbf{Q}}$, where $\tilde{\mathbf{Q}}$ is independent of \mathbf{h} and it is associated with the vertical hamiltonian vector field \mathbf{v} , i.e. we have

$$\mathbf{d}\tilde{\mathbf{Q}} = i_{\mathbf{v}} \omega = (-1)^{n-q-1} \mathbf{v}_A \wedge \mathbf{d}\mathbf{y}^A - \mathbf{d}\mathbf{p}_A \wedge \mathbf{v}^A - \nu. \quad (31)$$

If we employ notation $\mathbf{z}^I = (y_{(\mu)}^A, p_{A(\mu)^{q+1}})$ then we can write

$$\tilde{\mathbf{Q}} = \mathbf{q} + \mathbf{d}\mathbf{z}^{I_1} \wedge \mathbf{q}_{I_1} + \cdots + \mathbf{d}\mathbf{z}^{(I)_{n-1}} \wedge \mathbf{q}_{(I)_{n-1}},$$

where $\mathbf{q}, \dots, \mathbf{q}_{(I)_{n-1}}$ are M -horizontal $(n-1), \dots, 0$ -forms on \mathbf{F} , respectively. We have

$$\mathbf{d}\mathbf{Q}_{n-1} = \mathbf{d}\mathbf{q} - \mathbf{d}\mathbf{z}^I \wedge (\mathbf{d}\mathbf{z}^J \wedge \mathbf{q}_{I,J} + \mathbf{d}^{\mathbf{H}} \mathbf{q}_I) + \cdots + (-1)^{n-1} \mathbf{d}\mathbf{z}^{(I)_{n-1}} \wedge (\mathbf{d}\mathbf{z}^{I_n} \wedge \mathbf{q}_{(I)_{n-1}, I_n} + \mathbf{d}^{\mathbf{H}} \mathbf{q}_{(I)_{n-1}})$$

where $\mathbf{d}^{\mathbf{H}} \mathbf{q} = \mathbf{d}x^{\mu} \wedge \mathfrak{L}_{\partial_{\mu}} \mathbf{q}$ is horizontal part of the exterior derivative operator \mathbf{d} , and (31) yields $\mathbf{Q}_{n-1}(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0$ for all vertical vectors $\forall \mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbf{VF}$. This is possible only if on every fibre $\mathbf{F}_{\mathbf{x}} = \mathbb{P}_{\mathbf{M}, \mathbf{F}}^{-1}(\mathbf{x})$ there exist locally, or if $\mathbf{F}_{\mathbf{x}}$ is contractible then globally, M -horizontal $\tilde{\mathbf{q}}_{(I)_{n-2}}$ such that $\mathbf{q}_{(I)_{n-1}} = (-1)^{n-2} (n-1) \tilde{\mathbf{q}}_{[(I)_{n-2}, I_{n-1}]}$. Thus, we can write

$$\begin{aligned} \mathbf{Q}_{n-1} &= \mathbf{q} + \mathbf{d}\mathbf{z}^{I_1} \wedge \mathbf{q}_{I_1} + \cdots + \mathbf{d}\mathbf{z}^{(I)_{n-2}} \wedge \mathbf{q}_{(I)_{n-2}} - (-1)^{n-2} \mathbf{d}\mathbf{z}^{(I)_{n-2}} \wedge \mathbf{d}^{\mathbf{H}} \tilde{\mathbf{q}}_{(I)_{n-2}} + \mathbf{d}(\mathbf{d}\mathbf{z}^{(I)_{i-2}} \wedge \tilde{\mathbf{q}}_{(I)_{n-2}}) \\ &= \mathbf{Q}_{n-2} + \mathbf{d}(\mathbf{d}\mathbf{z}^{(I)_{i-2}} \wedge \tilde{\mathbf{q}}_{(I)_{n-2}}). \end{aligned}$$

If we continue in calculation of \mathbf{Q}_i 's until \mathbf{Q}_0 then we get $\tilde{\mathbf{Q}} = \mathbf{Q}_0 + \mathbf{d}\alpha$, where $\mathbf{Q} = \mathbf{Q}_0$ is \mathbf{M} -horizontal $(n-1)$ -form, keeping

$$i_{\mathbf{v}} \omega = (-1)^{n-q-1} \mathbf{v}_A \wedge \mathbf{d}\mathbf{y}^A - \mathbf{d}\mathbf{p}_A \wedge \mathbf{v}^A - \nu = \mathbf{d}\mathbf{Q}.$$

Now, recall the proof of theorem II.1 we see that \mathbf{Q} is simply differentiable and we also get

$$\begin{aligned} \mathbf{v}_A &= \frac{\partial^R \mathbf{Q}}{\partial \mathbf{y}^A}, \\ \mathbf{v}^A &= -\frac{\partial^L \mathbf{Q}}{\partial \mathbf{p}_A}, \\ \nu &= -\mathbf{d}^M \mathbf{Q}. \end{aligned} \tag{32}$$

□

The local Poisson brackets among two \mathbf{M} -horizontal observables $\mathbf{Q}_1, \mathbf{Q}_2$ can be expressed by using the relations (32) as

$$\llbracket \mathbf{Q}_1, \mathbf{Q}_2 \rrbracket = \frac{\partial^R \mathbf{Q}_1}{\partial \mathbf{y}^A} \wedge \frac{\partial^L \mathbf{Q}_2}{\partial \mathbf{p}_A} - \frac{\partial^R \mathbf{Q}_2}{\partial \mathbf{y}^A} \wedge \frac{\partial^L \mathbf{Q}_1}{\partial \mathbf{p}_A}.$$

This expression mimics the well known formula of the classical mechanics.

IV. EINSTEIN-CARTAN THEORY

A. Configuration Bundle of Einstein-Cartan Theory

In the theory of General Relativity or Einstein-Cartan theory the gravitational interaction is described through evolution of the geometry of the spacetime $\mathbf{M} = \Sigma \times \mathbb{R}$. Therefore we can consider the spacetime only as a topological manifold. For simplicity we assume that Σ is compact oriented boundaryless three dimensional manifold.

Let Σ_0 be given embedding of Σ in \mathbf{M} . Σ_0 splits \mathbf{M} into two parts. We choose one and denote its closure as $\mathbf{M}_{\Sigma_0}^+$. Closure of the complementar set is denoted as $\mathbf{M}_{\Sigma_0}^-$. $\bar{\mathbf{M}}_{\Sigma_0}^\pm$ denote one point compactifications of $\mathbf{M}_{\Sigma_0}^\pm$ and if $\pm\infty$ are added points called future and past of \mathbf{M} then two point compactification of \mathbf{M} is defined as $\bar{\mathbf{M}} = \{\pm\infty\} \cup \mathbf{M}$. Diffeomorphism $\mathfrak{s}_{\mathbf{M}} : \Sigma \times \mathbb{R} \rightarrow \mathbf{M}$ defines for each $t \in \mathbb{R}$ an embedding $t : \Sigma \rightarrow \mathbf{M}$ by $t(\mathbf{x}_\Sigma) = \mathfrak{s}_{\mathbf{M}}(\mathbf{x}_\Sigma, t)$, where $\mathbf{x}_\Sigma \in \Sigma$. If $\lim_{t \rightarrow \pm\infty} \mathfrak{s}(\mathbf{x}_\Sigma, t) = \pm\infty$ then the map $\mathfrak{s}_{\mathbf{M}}$ is called slicing of \mathbf{M} . We say that curve γ on \mathbf{M} is topologically causal, or t-causal, if there exists a slicing $\mathfrak{s}_{\mathbf{M}}$ of \mathbf{M} such that the curve γ intersects every $\Sigma_t = t(\Sigma)$ just once. t-causal curve γ is called future oriented if $\lim_{s \rightarrow \pm\infty} \gamma(s) = \pm\infty$, where s is a parameter on γ .

Let \mathbf{E} denote a graded bundle of right-handed coframes over \mathbf{M} , i.e. the fibre bundle $\mathbb{p}_{\mathbf{M}, \mathbf{E}} : \mathbf{E} \rightarrow \mathbf{M}$ of right-handed basis on the cotangent bundle $T^*\mathbf{M}$ defined by

$$\mathbf{E} = \bigcup_{\mathbf{x}, \mathbf{e}} \mathbf{e}_{\mathbf{x}},$$

where $\mathbf{e}_{\mathbf{x}} = (\mathbf{x}, \mathbf{e}^a)$ is right-handed base of the cotangent space $T_{\mathbf{x}}^*\mathbf{M}$. Its typical fibre $\mathbf{E}^f \simeq \mathbf{E}_{\mathbf{x}} = \mathbb{p}_{\mathbf{M}, \mathbf{E}}^{-1}(\mathbf{x})$ is diffeomorphic with the positive general linear group $\text{GL}^+(\mathbb{R}, 4) = \{g \in \text{GL}^+(\mathbb{R}, 4), \det(g) > 0\}$. In addition, we also assume that the topological shapes of Σ are restricted in such a way that the bundle \mathbf{E} is trivial. General case will be discussed in the forthcoming parts of the series.

In the Einstein-Cartan theory $\mathbf{e} \in \mathbf{E}$ is interpreted as an orthonormal coframe settled in the point $\mathbf{x} = \mathbb{p}_{\mathbf{M}, \mathbf{E}}(\mathbf{e})$. Since the orthonormal coframe $\mathbf{e} = (\mathbf{x}, \mathbf{e}^a)$ consists of \mathbf{M} -horizontal forms \mathbf{e}^a and $T_{\mathbf{x}}\mathbf{M} = \mathbb{p}_{\mathbf{M}, \mathbf{E}*} T_{\mathbf{e}}\mathbf{E}$ this defines a metric $\mathbf{g}^{(\mathbf{e})}$ with Minkowski signature on $T_{\mathbf{x}}\mathbf{M}$ by

$$\mathbf{g}^{(\mathbf{e})}(\mathbb{p}_{\mathbf{M}, \mathbf{E}*} \mathbf{v}, \mathbb{p}_{\mathbf{M}, \mathbf{E}*} \mathbf{w}) = \eta_{ab} \mathbf{e}^a(\mathbb{p}_{\mathbf{M}, \mathbf{E}*} \mathbf{v}) \mathbf{e}^b(\mathbb{p}_{\mathbf{M}, \mathbf{E}*} \mathbf{w}), \tag{33}$$

where $\mathbf{v}, \mathbf{w} \in T_{\mathbf{e}}\mathbf{E}$. We can also consider oriented area and volume forms $\omega_{ab}^{(\mathbf{e})}$ and $\omega_a^{(\mathbf{e})}$ in $\mathbf{x} \in \mathbf{M}$

$$\begin{aligned}\omega_{ab}^{(\mathbf{e})}(\mathbb{p}_{\mathbf{M}, \mathbf{E}*}\mathbf{v}, \mathbb{p}_{\mathbf{M}, \mathbf{E}*}\mathbf{w}) &= \left(\frac{1}{2}\varepsilon_{abcd}\mathbf{e}^c \wedge \mathbf{e}^d\right)(\mathbf{v}, \mathbf{w}), \\ \omega_a^{(\mathbf{e})}(\mathbb{p}_{\mathbf{M}, \mathbf{E}*}\mathbf{u}, \mathbb{p}_{\mathbf{M}, \mathbf{E}*}\mathbf{v}, \mathbb{p}_{\mathbf{M}, \mathbf{E}*}\mathbf{w}) &= \left(\frac{1}{3!}\varepsilon_{abcd}\mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d\right)(\mathbf{u}, \mathbf{v}, \mathbf{w}),\end{aligned}\tag{34}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in T_{\mathbf{e}}\mathbf{E}$.

Symmetric bilinear form $\eta_{ab}\mathbf{e}^a \otimes \mathbf{e}^b$ on $T_{\mathbf{e}}\mathbf{E}$ is invariant under proper Lorentz transformations $(\mathbf{x}, \mathbf{e}^a) \rightarrow (\mathbf{x}, O^a_{\bar{a}}(\mathbf{e})\mathbf{e}^{\bar{a}})$, i.e. $\det O^a_{\bar{a}}(\mathbf{e}) = 1$ and $\mathbf{x}, O^0_0(\mathbf{e}) > 0$. Such transformation can be extended into d-jet diffeomorphism if and only if the matrix $O^a_{\bar{a}}(\mathbf{e})$ depends only on \mathbf{x} , i.e. $O^a_{\bar{a}}(\mathbf{e}) = O^a_{\bar{a}}(\mathbf{x})$. Group of such local (proper) Lorentz transformations is denoted by $SO(\boldsymbol{\eta}, \mathbf{M})$.

In the standard theory of General Relativity the parallel transport is given by Riemann-Levi-Civita connection, on the other hand the Einstein-Cartan theory works with a general metric-compatible connection which should be considered as independent variable. If $\mathbf{e} \in \mathbf{E}$ is orthonormal coframe of the point $\mathbf{x} = \mathbb{p}_{\mathbf{M}, \mathbf{E}}(\mathbf{e}) \in \mathbf{M}$ then the covariant derivative operator ∇ of the metric-compatible connection is given by $\nabla_{\mathbf{v}}\mathbf{e}^a = -i_{\mathbf{v}}\mathbf{A}^a_b\mathbf{e}^b$ on small neighborhood of \mathbf{x} , where $\mathbf{v} \in T_{\mathbf{x}}\mathbf{M}$ and $\mathbf{A}^{ab} = -\mathbf{A}^{ba} = \mathbf{A}^a_c\eta^{bc}$ is the connection 1-form.

Let $\mathbb{p}_{\mathbf{E}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{E}$ denote a fibre bundle with coordinates $(\mathbf{x}, \mathbf{e}^a, \mathbf{A}^{ab})$. Since the bundle $\mathbf{E} \rightarrow \mathbf{M}$ is trivial under our assumptions the bundle $\mathbb{p}_{\mathbf{M}, \mathbf{E}} \circ \mathbb{p}_{\mathbf{E}, \mathbf{Y}} = \mathbb{p}_{\mathbf{M}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{M}$ is also trivial. Bundle $\mathbb{p}_{\mathbf{M}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{M}$ is obviously graded manifold over \mathbf{M} .

Relation $\nabla_{\mathbf{v}}\mathbf{e}^a = -i_{\mathbf{v}}\mathbf{A}^a_b\mathbf{e}^b$ yields that the action of the local Lorentz transformation $O \in SO(\boldsymbol{\eta}, \mathbf{M})$ on \mathbf{Y} is given by

$$O : (x^\mu, \mathbf{e}^a, \mathbf{A}^{ab}) \mapsto (x^\mu, O^a_{\bar{a}}\mathbf{e}^{\bar{a}}, O^a_{\bar{a}}O^b_{\bar{b}}\mathbf{A}^{\bar{a}\bar{b}} + O^a_{\bar{a}}\eta^{\bar{a}\bar{b}}dO^b_{\bar{b}}).\tag{35}$$

Let us consider a space $\Lambda(\mathbf{Y}, T^{\otimes}\mathbf{M})$ of forms on \mathbf{Y} with values in the total tensor space $T^{\otimes}\mathbf{M}$ over \mathbf{M} . We can define an exterior $SO(\boldsymbol{\eta}, \mathbf{M})$ -covariant derivative \mathbf{D} on $\Lambda(\mathbf{Y}, T^{\otimes}\mathbf{M})$ by

$$\mathbf{D}\mathbf{u}^a = d\mathbf{u}^a + \mathbf{A}^a_b \wedge \mathbf{u}^b,$$

where $\mathbf{u}^a \in \Lambda(\mathbf{Y}, T\mathbf{M})$. Relation $\mathbf{D}\mathbf{D}\mathbf{u}^a = \mathbf{D}\mathbf{A}^a_b \wedge \mathbf{u}^b$ yields the expression

$$\mathbf{D}\mathbf{A}^a_b = d\mathbf{A}^a_b + \mathbf{A}^a_c \wedge \mathbf{A}^c_b$$

of the curvature and Bianchi identity $\mathbf{D}\mathbf{D}\mathbf{A}^a_b = 0$ is also satisfied. It is clear that $\mathbf{D}\mathbf{A}^{ab}$ transforms as a tensor under the action (35) of the local Lorentz group $SO(\boldsymbol{\eta}, \mathbf{M})$.

The graded bundle $\mathbb{p}_{\mathbf{M}, \mathbf{Y}} : \mathbf{Y} \rightarrow \mathbf{M}$ is called full configuration bundle of the Einstein-Cartan theory. Its d-jet dual $\mathbf{J}\mathbf{Y}^*$ is equipped with the first canonical form

$$\theta^{(\mathbf{Y})} = -\mathbf{p}_a \wedge d\mathbf{e}^a - \frac{1}{2}\mathbf{p}_{ab} \wedge d\mathbf{A}^{ab} + \mathbf{h}\tag{36}$$

and the multisymplectic form

$$\omega^{(\mathbf{Y})} = -d\theta^{(\mathbf{Y})} = d\mathbf{p}_a \wedge d\mathbf{e}^a + \frac{1}{2}d\mathbf{p}_{ab} \wedge d\mathbf{A}^{ab} - d\mathbf{h},\tag{37}$$

where $\mathbf{p}_a, \mathbf{p}_{ab}$ are canonical momenta related to \mathbf{e}^a and \mathbf{A}^{ab} and \mathbf{h} is affine hamiltonian coordinate.

B. Equations of Motion, Covariant Legendre Transform and Reduced Phase Space Bundles

Let \mathbf{M}_I be compact portion of the spacetime bounded by two disjoint embeddings Σ_{ini} and Σ_{fin} and let $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{Y})$ be the section of the configuration bundle \mathbf{Y} such that Σ_{ini} and Σ_{fin} are Cauchy surfaces and $\varphi^* \mathbf{e}^a|_{\partial \mathbf{M}_I}$ are future oriented coframes, i.e. for any t-causal curve γ transverse to $\partial \mathbf{M}_I$ there holds $\mathbf{e}^0(\dot{\gamma}) > 0$, then the Einstein-Hilbert-Palatini action of the Einstein-Cartan theory is given by

$$\mathbf{S} = \int_{\mathbf{M}_I} (\mathbf{j}\varphi)^* \mathbf{L} = \int_{\mathbf{M}_I} \frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{D}_\varphi \mathbf{A}_\varphi^{ab} \wedge \mathbf{e}_\varphi^c \wedge \mathbf{e}_\varphi^d, \quad (38)$$

where κ is a Newton's constant (the speed of light c is set to 1), $\mathbf{e}_\varphi^a = \varphi^* \mathbf{e}^a$, $\mathbf{A}_\varphi^{ab} = \varphi^* \mathbf{A}^{ab}$ and \mathbf{D}_φ means $\text{SO}(\eta, \mathbf{M})$ -covariant exterior derivative on \mathbf{M} given by the connection \mathbf{A}_φ^{ab} . Variation of the action (38)

$$0 = \int_{\mathbf{M}_I} \delta \mathbf{e}^a \wedge \left(\frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}_\varphi \mathbf{A}_\varphi^{bc} \wedge \mathbf{e}_\varphi^d \right) + \int_{\mathbf{M}_I} \frac{1}{2} \delta \mathbf{A}^{ab} \wedge \left(-\frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{e}_\varphi^c \wedge \mathbf{D}_\varphi \mathbf{e}_\varphi^d \right),$$

where $\delta \mathbf{e}^a|_{\partial \mathbf{M}_I} = \delta \mathbf{A}^{ab}|_{\partial \mathbf{M}_I} = 0$, yields equations of motion

$$0 = \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}_\varphi \mathbf{A}_\varphi^{bc} \wedge \mathbf{e}_\varphi^d, \quad (39)$$

$$0 = -\frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{e}_\varphi^c \wedge \mathbf{D}_\varphi \mathbf{e}_\varphi^d. \quad (40)$$

The second equation implies that the torsion $\mathbf{D}_\varphi \mathbf{e}_\varphi^a$ of the connection \mathbf{A}_φ^{ab} is vanishing on $(\mathbf{M}, \mathbf{g}_\varphi = \eta_{ab} \mathbf{e}_\varphi^a \otimes \mathbf{e}_\varphi^b)$ therefore the first one is equivalent to the Einstein equations of the General Relativity.

In section II we made assumption that considered Lagrangians are regular, but the Einstein-Hilbert-Palatini Lagrangian \mathbf{L} given by (38) depends at most linearly on the generalized velocity $\mathbf{d}\mathbf{A}^{ab}$, hence the Lagrangian \mathbf{L} is singular and we can not express any velocity through the canonical momenta. Instead of exploring the general case of singular Lagrangians we use the following construction. The Legendre transformation (6) $\mathbb{F}\mathbf{L} : \mathbf{J}\mathbf{Y} \rightarrow \mathbf{J}\mathbf{Y}^*$ yields

$$\mathbf{p}_a = 0, \quad (41)$$

$$\mathbf{p}_{ab} = \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{e}^c \wedge \mathbf{e}^d, \quad (42)$$

$$\mathbf{H} = -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{A}_{\bar{a}}^a \wedge \mathbf{A}^{\bar{a}b} \wedge \mathbf{e}^c \wedge \mathbf{e}^d. \quad (43)$$

The dynamical Cartan-Poincaré form on $\mathbf{P} = \mathbb{F}\mathbf{L}(\mathbf{J}\mathbf{Y}) \simeq \mathbf{Y}$ is given by the first equality in (7)

$$\Theta = \theta^{(\mathbf{Y})}|_{\mathbf{P}} = -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \quad (44)$$

or if we use embedding $\mathbb{F}\mathbf{H} : \mathbf{P} \rightarrow \mathbf{J}\mathbf{E}^*$

$$\mathbf{e}^a = \mathbf{e}^a, \quad (45)$$

$$\mathbf{G}_a = \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{A}^{bc} \wedge \mathbf{e}^d, \quad (46)$$

$$\mathbf{h} = \mathbf{H}, \quad (47)$$

then

$$\Theta = -\frac{1}{2}\mathbf{e}^a \wedge d\mathbf{G}_a - \frac{1}{2}\mathbf{G}_a \wedge d\mathbf{e}^a + \mathbf{H}(\mathbf{e}, \mathbf{G}), \quad (48)$$

where the Hamiltonian $\mathbf{H}(\mathbf{e}, \mathbf{G})$ is given by the expression, see convention of the Hodge operator $*$ in¹⁹ 5.8.1,

$$\mathbf{H} \equiv \mathbf{H}(\mathbf{e}, \mathbf{G}) = 4\pi\kappa \left[(\mathbf{G}_a \wedge \mathbf{e}^b) \wedge *(\mathbf{G}_b \wedge \mathbf{e}^a) - \frac{1}{2}(\mathbf{G}_a \wedge \mathbf{e}^a) \wedge *(\mathbf{G}_b \wedge \mathbf{e}^b) \right]. \quad (49)$$

The dynamical multisymplectic form is given by

$$\Omega = -d\Theta = -\frac{1}{16\pi\kappa}\varepsilon_{abcd}\mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge d\mathbf{e}^d = d\mathbf{G}_a \wedge d\mathbf{e}^a - d\mathbf{H}. \quad (50)$$

This construction shows that the multisymplectic reduction of \mathbf{JY}^* defined by (41) and (42) yields that we can use the d-jet dual \mathbf{JE}^* of the graded manifold of right-handed coframes \mathbf{E} as the reduced kinematical phase space of the Einstein-Cartan theory if the Cartan-Poincaré form on \mathbf{JE}^* is given by

$$\theta = -\frac{1}{2}\mathbf{e}^a \wedge d\mathbf{G}_a - \frac{1}{2}\mathbf{G}_a \wedge d\mathbf{e}^a + \mathbf{h} = -\mathbf{G}_a \wedge d\mathbf{e}^a + \mathbf{h} + d\left(\frac{1}{2}\mathbf{G}_a \wedge \mathbf{e}^a\right) \quad (51)$$

instead of the first canonical form defined for \mathbf{JE}^* by (4). These two forms differ by exact term $d(\frac{1}{2}\mathbf{G}_a \wedge \mathbf{e}^a)$ therefore the kinematical multisymplectic form of the Einstein-Cartan theory given by

$$\omega = -d\theta = d\mathbf{G}_a \wedge d\mathbf{e}^a - d\mathbf{h}. \quad (52)$$

and the second canonical form (5) coincide.

Lemma III.4 yields that following form fields are observables with hamiltonian vectors on \mathbf{JE}^*

$$\begin{aligned} \mathbf{e}(\gamma) &= \gamma_a \wedge \mathbf{e}^a \leftrightarrow \mathbf{v}(\gamma) = \gamma_a \partial_{\mathbf{G}_a} - (d\gamma_a \wedge \mathbf{e}^a) \partial_{\mathbf{h}}, \\ \mathbf{G}(\mathbf{E}) &= \mathbf{G}_a \wedge \mathbf{E}^a \leftrightarrow \mathbf{v}(\mathbf{E}) = -\mathbf{E}^a \partial_{\mathbf{e}^a} - (\mathbf{G}_a \wedge d\mathbf{E}^a) \partial_{\mathbf{h}}, \end{aligned}$$

where γ_a and \mathbf{E}^a are smearing \mathbf{M} -horizontal 2- and 1-forms, respectively, depending only on the spacetime coordinates x^μ and their non-vanishing local covariant Poisson brackets are

$$\{\{\mathbf{e}(\gamma), \mathbf{G}(\mathbf{E})\}\} = \gamma_a \wedge \mathbf{E}^a.$$

There also exist smeared observables related to 2-area and 3-volume. Indeed, it is easy to show that following objects are hamiltonian pairs

$$\begin{aligned} \omega^{(3)}(\lambda) &= \frac{1}{3!}\varepsilon_{abcd}\lambda^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d \leftrightarrow \mathbf{v}(\lambda) = \left(\frac{1}{2}\varepsilon_{abcd}\lambda^a \mathbf{e}^c \wedge \mathbf{e}^d\right)\partial_{\mathbf{G}_b} - \left(\frac{1}{3!}\varepsilon_{abcd}d\lambda^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d\right)\partial_{\mathbf{h}}, \\ \omega^{(2)}(\mathbf{B}) &= \frac{1}{4}\varepsilon_{abcd}\mathbf{B}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \leftrightarrow \mathbf{v}(\mathbf{B}) = \left(\frac{1}{2}\varepsilon_{abcd}\mathbf{B}^{ab} \wedge \mathbf{e}^c\right)\partial_{\mathbf{G}_d} - \left(\frac{1}{4}\varepsilon_{abcd}d\mathbf{B}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d\right)\partial_{\mathbf{h}}, \end{aligned}$$

where λ^a , $\mathbf{B}^{ab} = -\mathbf{B}^{ba}$ are horizontal 0-, 1-forms, respectively, depending only on spacetime coordinates x^μ .

C. Gauge group of the Einstein-Cartan Theory

Now, we want to explore the symmetry group of the Einstein-Cartan theory. We employ the coordinates $\mathbf{e}^a, \mathbf{A}^{ab}$ rather than the canonical pair $\mathbf{e}^a, \mathbf{G}_a$. The first observation is that the Lagrangian \mathbf{L} of the Einstein-Cartan theory is invariant under the action (35) of the local Lorentz group $\text{SO}(\boldsymbol{\eta}, \mathbf{M})$. Its infinitesimal action is given by

$$\begin{aligned}\bar{\mathbf{e}}^a &= \mathbf{e}^a - \Lambda^a_b \mathbf{e}^b \\ \bar{\mathbf{A}}^{ab} &= \mathbf{A}^{ab} - \Lambda^a_{\bar{a}} \mathbf{A}^{\bar{a}b} - \Lambda^b_{\bar{b}} \mathbf{A}^{a\bar{b}} + \mathbf{d}\Lambda^{ab} = \mathbf{A}^{ab} + \mathbf{D}\Lambda^{ab},\end{aligned}\tag{53}$$

where $\Lambda^{ab} = -\Lambda^{ba}$ are 0-forms on \mathbf{M} playing role of the coordinates on the algebra of the local Lorentz group $\text{SO}(\boldsymbol{\eta}, \mathbf{M})$. The generating vector of the infinitesimal action (53) is given by

$$\mathbf{v}(\Lambda) = -\Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a} + \frac{1}{2} \mathbf{D}\Lambda^{ab} \partial_{\mathbf{A}^{ab}}\tag{54}$$

Its basic properties are summarized in the following lemma.

Lemma IV.1.

- i) *The dynamical Cartan-Poincaré form (44) is invariant under the action (35) of the local Lorentz group $\text{SO}(\boldsymbol{\eta}, \mathbf{M})$ therefore*

$$\mathfrak{L}_{\mathbf{v}(\Lambda)} \Theta = 0.$$

- ii) *Noether's current $\mathbf{t}(\Lambda) := i_{\mathbf{v}(\Lambda)} \Theta$ is given by formula*

$$\mathbf{t}(\Lambda) := -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{D}\Lambda^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d.$$

- iii) *Vanishing of the Noether's charge*

$$\mathbf{T}_{\Sigma_E}^\varphi(\Lambda) = \int_{\Sigma_E} \varphi^* \mathbf{t}(\Lambda) = 0$$

for all embeddings $\Sigma_E \simeq \Sigma$ and all $\Lambda^{ab} \in \mathfrak{alg}(\text{SO}(\boldsymbol{\eta}, \mathbf{M}))$ implies that $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is a solution of the Hamilton equation (40).

Proof. Let $\mathbf{O}(s\Lambda) \in \text{SO}(\boldsymbol{\eta}, \mathbf{M})$ denote one parametric group associated to the vector $\mathbf{v}(\Lambda)$. Direct calculation of its action on Θ yields

$$\begin{aligned}\bar{\Theta} &= -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \bar{\mathbf{D}}\bar{\mathbf{A}}^{ab} \wedge \bar{\mathbf{e}}^c \wedge \bar{\mathbf{e}}^d \\ &= -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{O}^a_{\bar{a}}(s\Lambda) \mathbf{O}^b_{\bar{b}}(s\Lambda) \mathbf{O}^c_{\bar{c}}(s\Lambda) \mathbf{O}^d_{\bar{d}}(s\Lambda) \mathbf{D}\mathbf{A}^{\bar{a}\bar{b}} \wedge \mathbf{e}^{\bar{c}} \wedge \mathbf{e}^{\bar{d}} \\ &= -\det(\mathbf{O}^{\bar{a}}_{\bar{b}}(s\Lambda)) \frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d = \Theta,\end{aligned}$$

since $\det(\mathbf{O}^{\bar{a}}_{\bar{b}}(s\Lambda)) = 1$. Therefore we have

$$\mathfrak{L}_{\mathbf{v}(\Lambda)} \Theta = \lim_{s \rightarrow 0} \frac{\bar{\Theta} - \Theta}{s} = 0.$$

The interior product of $\mathbf{v}(\Lambda)$ with the Cartan-Poincaré form Θ is

$$\mathbf{t}(\Lambda) = i_{\mathbf{v}(\Lambda)}\Theta = -\frac{1}{32\pi\kappa}\varepsilon_{abcd}\mathbf{D}\Lambda^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d$$

and the integration over given emmbeding Σ_E yields relation

$$\begin{aligned} \mathbf{T}_{\Sigma_E}^\varphi(\Lambda) &= \int_{\Sigma_E} \varphi^* \mathbf{t}(\Lambda) = \int_{\Sigma_E} -\frac{1}{32\pi\kappa}\varepsilon_{abcd}\mathbf{D}_\varphi\Lambda^{ab} \wedge \mathbf{e}_\varphi^c \wedge \mathbf{e}_\varphi^d \\ &= \int_{\Sigma_E} \frac{1}{2}\Lambda^{ab} \left(\frac{1}{8\pi\kappa}\varepsilon_{abcd}\mathbf{e}_\varphi^c \wedge \mathbf{D}_\varphi\mathbf{e}_\varphi^d \right) + \oint_{\partial\Sigma_E} -\frac{1}{32\pi\kappa}\varepsilon_{abcd}\Lambda^{ab}\mathbf{e}_\varphi^c \wedge \mathbf{e}_\varphi^d. \end{aligned}$$

The last term is vanishing since $\partial\Sigma_E = \emptyset$. The condition $\mathbf{T}_{\Sigma_E}^\varphi(\Lambda) = 0$ for all Λ^{ab} implies that $-\frac{1}{8\pi\kappa}\varepsilon_{abcd}\mathbf{e}_\varphi^c \wedge \mathbf{D}_\varphi\mathbf{e}_\varphi^d = 0$ on the embedding Σ_E and arbitrariness of the embedding yields that the equation (40) is satisfied on whole \mathbf{M} and vice versa if the equation (40) is satisfied then $\mathbf{T}_{\Sigma_E}^\varphi(\Lambda) = 0$ for all embeddings Σ_E and all Λ^{ab} . \square

Einstein's principle²⁰ of General Relativity says that the laws of the Nature are independent on the choice of the spacetime coordinates we are using for the description of the physical system. Therefore the action (38) should be invariant under the action of the group of diffeomorphisms $\mathcal{D}\text{iff}(\mathbf{M})$. Let $\eta_{\mathbf{M}}(s) \in \mathcal{D}\text{iff}(\mathbf{M})$ denote one parameter group generated by the spacetime vector $\xi \in \mathbf{TM}$. Its prolongation $\eta_{\mathbf{P}}(s)$ to the fibre bundle $\mathbf{P} \rightarrow \mathbf{M}$ is given by

$$\eta_{\mathbf{P}} : (x^\mu, \mathbf{e}^a, \mathbf{A}^{ab}) \mapsto (\eta_{\mathbf{M}}^\mu, (\eta_{\mathbf{M}}^{-1})^* \mathbf{e}^a, (\eta_{\mathbf{M}}^{-1})^* \mathbf{A}^{ab}). \quad (55)$$

The generating vector $\mathbf{w}(\xi)$ of $\eta_{\mathbf{P}}$ on \mathbf{P} is given by an expression

$$\mathbf{w}(\xi) = \xi - i_{\mathbf{d}\xi}\mathbf{e}^a \partial_{\mathbf{e}^a} - \frac{1}{2}i_{\mathbf{d}\xi}\mathbf{A}^{ab} \partial_{\mathbf{A}^{ab}}. \quad (56)$$

We have similar lemma for $\mathbf{w}(\xi)$ as for $\mathbf{v}(\Lambda)$.

Lemma IV.2.

- i) *The dynamical Cartan-Poincaré form (44) is invariant under the action (55) of the diffeomorphism group $\mathcal{D}\text{iff}(\mathbf{M})$ therefore*

$$\mathbf{L}_{\mathbf{w}(\xi)}\Theta = 0.$$

- ii) *Noether's current $\mathbf{r}(\xi) := i_{\mathbf{w}(\xi)}\Theta$ is given by formula*

$$\mathbf{r}(\xi) := \frac{1}{16\pi\kappa}\varepsilon_{abcd}\mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c i_\xi \mathbf{e}^d + \frac{1}{16\pi\kappa}\varepsilon_{abcd}i_\xi \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{D}\mathbf{e}^d + \mathbf{d} \left(\frac{1}{32\pi\kappa}\varepsilon_{abcd}i_\xi \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \right)$$

- iii) *If for the section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ the Hamilton equation (40) is satisfied then the vanishing of the Noether's charge*

$$\mathbf{R}_{\Sigma_E}^\varphi(\xi) = \int_{\Sigma_E} \varphi^* \mathbf{r}(\xi) = 0$$

for all embeddings $\Sigma_E \simeq \Sigma$ and all $\xi \in \text{alg}(\mathcal{D}\text{iff}(\mathbf{M}))$ implies that $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is also a solution of the Hamilton equation (39).

Proof. At first we need to calculate some auxiliar relations. We employ notation $\mathbf{y}^A = (\mathbf{e}^a, \mathbf{A}^{ab})$. Cartan-Poincaré form Θ can be written as

$$\Theta = \mathbf{d}y_\mu^A \wedge \Theta_A^{\mu\nu} \mathbf{d}\Sigma_\nu + \tilde{H} \mathbf{d}\Sigma,$$

where $\Theta_A^{\mu\nu}$ and \tilde{H} are certain functions on \mathbf{P} . Since Θ does not depend explicitly on the coordinate x^μ we have for $\xi = \xi^\mu \partial_\mu$ followig relation

$$\begin{aligned} i_\xi \mathbf{d}\Theta + \mathbf{d}i_\xi \Theta &= -\mathbf{d}y_\mu^A \wedge \mathbf{d}\Theta^{\mu\nu} \wedge i_\xi \mathbf{d}\Sigma_\nu - \mathbf{d}\tilde{H} \wedge i_\xi \mathbf{d}\Sigma + \\ &+ \mathbf{d}y_\mu^A \wedge \mathbf{d}\Theta^{\mu\nu} \wedge i_\xi \mathbf{d}\Sigma_\nu + \mathbf{d}y_\mu^A \wedge \Theta^{\mu\nu} \wedge \mathbf{d}(i_\xi \Sigma_\nu) + \\ &+ \mathbf{d}\tilde{H} \wedge i_\xi \mathbf{d}\Sigma + \tilde{H} \mathbf{d}(i_\xi \mathbf{d}\Sigma) \end{aligned}$$

or after simplification

$$i_\xi \mathbf{d}\Theta + \mathbf{d}i_\xi \Theta = i_{\mathbf{d}\xi} \Theta. \quad (57)$$

Similar calculation yields

$$i_\xi \mathbf{d}\mathbf{y}^A + \mathbf{d}i_\xi \mathbf{y}^A = i_{\mathbf{d}\xi} \mathbf{y}^A. \quad (58)$$

Now, we can prove the first indentity *i*) of the lemma

$$\begin{aligned} \mathbb{F}_{\mathbf{w}(\xi)} \Theta &= i_{\mathbf{w}(\xi)} \mathbf{d}\Theta + \mathbf{d}i_{\mathbf{w}(\xi)} \Theta \\ &= i_\xi \mathbf{d}\Theta - \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{D}\mathbf{e}^d + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge i_{\mathbf{d}\xi} \mathbf{e}^d + \\ &+ \mathbf{d}i_\xi \Theta + \frac{1}{32\pi\kappa} \mathbf{d} \left(\varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \right). \end{aligned}$$

The first auxiliar formula (57) yields

$$\mathbb{F}_{\mathbf{w}(\xi)} \Theta = \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{a\bar{a}} \wedge \mathbf{A}^b_{\bar{a}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d - \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{A}^d_{\bar{d}} \wedge \mathbf{e}^{\bar{d}}.$$

The first term can be rewritten as

$$\begin{aligned} \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{a\bar{a}} \wedge \mathbf{A}^b_{\bar{a}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d &= \frac{1}{16\pi\kappa} \varepsilon_{abcd} \frac{1}{4} \varepsilon^{a\bar{a}c\bar{c}d} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} i_{\mathbf{d}\xi} \mathbf{A}^{\hat{a}\hat{b}} \wedge \mathbf{A}^b_{\bar{a}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \\ &= \frac{1}{32\pi\kappa} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} (\delta_{\hat{b}}^{\bar{a}} \delta_{\hat{c}\hat{d}}^{\hat{c}\hat{d}} + \delta_{\hat{c}}^{\bar{a}} \delta_{\hat{d}\hat{b}}^{\hat{c}\hat{d}} + \delta_{\hat{d}}^{\bar{a}} \delta_{\hat{b}\hat{c}}^{\hat{c}\hat{d}}) i_{\mathbf{d}\xi} \mathbf{A}^{\hat{a}\hat{b}} \wedge \mathbf{A}^b_{\bar{a}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \quad (59) \\ &= \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{A}^d_{\bar{d}} \wedge \mathbf{e}^{\bar{d}} \end{aligned}$$

which yields

$$\mathbb{F}_{\mathbf{w}(\xi)} \Theta = 0.$$

Noether's current is given by

$$\begin{aligned} \mathbf{r}(\xi) &= i_{\mathbf{w}(\xi)} \Theta = i_\xi \Theta + \frac{1}{32\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \\ &= -\frac{1}{32\pi\kappa} \varepsilon_{abcd} i_\xi \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c i_\xi \mathbf{e}^d + \frac{1}{32\pi\kappa} \varepsilon_{abcd} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d. \end{aligned}$$

Now, if we use the second auxiliar relation (58) then the similar calculation as in (59) yields the result

$$\mathbf{r}(\xi) = \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}\mathbf{A}^{ab} \wedge \mathbf{e}^c i_\xi \mathbf{e}^d + \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_\xi \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{D}\mathbf{e}^d + \mathbf{d} \left(\frac{1}{32\pi\kappa} \varepsilon_{abcd} i_\xi \mathbf{A}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \right).$$

Noether's charge on embedding Σ_E for the section $\varphi \in \text{Sec}(\mathbf{M}, \mathbf{P})$ is

$$\mathbf{R}_{\Sigma_E}^\varphi(\xi) = \int_{\Sigma_E} \varphi^* \mathbf{r}(\xi) = \int_{\Sigma_E} \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{D}_\varphi \mathbf{A}_\varphi^{ab} \wedge \mathbf{e}_\varphi^c i_\xi \mathbf{e}_\varphi^d + \int_{\Sigma_E} \frac{1}{16\pi\kappa} \varepsilon_{abcd} i_\xi \mathbf{A}_\varphi^{ab} \wedge \mathbf{e}_\varphi^c \wedge \mathbf{D}_\varphi \mathbf{e}_\varphi^d.$$

If φ is the solution of the Hamilton equation (40) then the second term is vanishing and the arbitrariness of the embedding Σ_E and ξ implies that the vanishing of the charge is equivalent to the Hamilton equation (39). \square

As a direct consequence of these two lemmata we have that the mutual vanishing of the Noether's charges $\mathbf{T}(\Lambda)$ and $\mathbf{R}(\xi)$ is equivalent to the Hamilton equations (39) and (40). Therefore it seems that the symmetry group G of the Einstein-Cartan theory can be constructed from groups $\text{SO}(\eta, \mathbf{M})$ and $\mathfrak{Diff}(\mathbf{M})$. The first step is to show, that the linear span of $\mathfrak{alg}(\text{SO}(\eta, \mathbf{M}))$ and $\mathfrak{alg}(\mathfrak{Diff}(\mathbf{M}))$ is closed subalgebra in Lie algebra of vector fields $(\mathbf{TP}, \llbracket, \rrbracket)$.

Lemma IV.3. *Span $(\mathfrak{alg}(\text{SO}(\eta, \mathbf{M})), \mathfrak{alg}(\mathfrak{Diff}(\mathbf{M})))$ is a Lie algebra and the commutators of the generators are*

i)

$$\llbracket \mathbf{v}(\Lambda), \mathbf{v}(\Lambda') \rrbracket = \mathbf{v}(\Lambda \eta \Lambda' - \Lambda' \eta \Lambda),$$

ii)

$$\llbracket \mathbf{w}(\xi), \mathbf{w}(\xi') \rrbracket = \mathbf{w}(\xi \xi' - \xi' \xi),$$

iii)

$$\llbracket \mathbf{w}(\xi), \mathbf{v}(\Lambda) \rrbracket = \mathbf{v}(\xi \Lambda - \Lambda \xi).$$

Proof. We have for i) an expression

$$\llbracket \mathbf{v}(\Lambda), \mathbf{v}(\Lambda') \rrbracket = \llbracket -\Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a}, -\Lambda'^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \rrbracket + \left\llbracket \frac{1}{2} \mathbf{D} \Lambda^{ab} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{D} \Lambda'^{cd} \partial_{\mathbf{A}^{cd}} \right\rrbracket.$$

The first bracket can be evaluated as

$$\begin{aligned} \llbracket -\Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a}, -\Lambda'^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \rrbracket &= \Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a} \Lambda'^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} - \Lambda'^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a} \\ &= -(\Lambda^a_c \Lambda'^c_b - \Lambda'^a_c \Lambda^c_b) \mathbf{e}^b \partial_{\mathbf{e}^a}. \end{aligned}$$

Evaluation of the second term yields

$$\begin{aligned} \left\llbracket \frac{1}{2} \mathbf{D} \Lambda^{ab} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{D} \Lambda'^{cd} \partial_{\mathbf{A}^{cd}} \right\rrbracket &= \left\llbracket \frac{1}{2} \mathbf{d} \Lambda^{ab} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{d} \Lambda'^{cd} \partial_{\mathbf{A}^{cd}} \right\rrbracket + \left\llbracket \frac{1}{2} \mathbf{d} \Lambda^{ab} \partial_{\mathbf{A}^{ab}}, -\Lambda'^c_{\bar{c}} \mathbf{A}^{\bar{c}d} \partial_{\mathbf{A}^{cd}} \right\rrbracket + \\ &\quad + \left\llbracket -\Lambda^a_{\bar{a}} \mathbf{A}^{\bar{a}b} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{d} \Lambda'^{cd} \partial_{\mathbf{A}^{cd}} \right\rrbracket + \left\llbracket -\Lambda^a_{\bar{a}} \mathbf{A}^{\bar{a}b} \partial_{\mathbf{A}^{ab}}, -\Lambda'^c_{\bar{c}} \mathbf{A}^{\bar{c}d} \partial_{\mathbf{A}^{cd}} \right\rrbracket \\ &= -\mathbf{d} \Lambda^{cb} \Lambda'^a_c \partial_{\mathbf{A}^{ab}} + \mathbf{d} \Lambda'^{cb} \Lambda^a_c \partial_{\mathbf{A}^{ab}} - (\Lambda^a_c \Lambda'^c_{\bar{c}} - \Lambda'^a_c \Lambda^c_{\bar{c}}) \mathbf{A}^{\bar{c}b} \partial_{\mathbf{A}^{ab}} \\ &= \frac{1}{2} \mathbf{d} (\Lambda^a_c \Lambda'^{cb} - \Lambda'^a_c \Lambda^{cb}) \partial_{\mathbf{A}^{ab}} - (\Lambda^a_c \Lambda'^c_{\bar{c}} - \Lambda'^a_c \Lambda^c_{\bar{c}}) \mathbf{A}^{\bar{c}b} \partial_{\mathbf{A}^{ab}} \\ &= \frac{1}{2} \mathbf{D} (\Lambda^a_c \Lambda'^{cb} - \Lambda'^a_c \Lambda^{cb}) \partial_{\mathbf{A}^{ab}}. \end{aligned}$$

These two relation establish the first equality *i)* of the lemma.

In order to prove the second relationship *ii)* we employ again for a while the notation $\mathbf{y}^A = (\mathbf{e}^a, \mathbf{A}^{ab})$. The generator $\mathbf{w}(\xi)$ can be rewritten as

$$\mathbf{w}(\xi) = \xi - i_{\mathbf{d}\xi} \mathbf{y}^A \partial_{\mathbf{y}^A}.$$

Thus, we have

$$\begin{aligned} \llbracket \mathbf{w}(\xi), \mathbf{w}(\xi') \rrbracket &= \xi_{\xi} \xi' - \xi^{\nu} \xi'^{\lambda}_{,\mu\nu} y_{\lambda}^A \partial_{y_{\mu}^A} + \xi'^{\nu} \xi^{\lambda}_{,\mu\nu} y_{\lambda}^A \partial_{y_{\mu}^A} \\ &\quad - \xi^{\nu}_{,\lambda} \xi'^{\lambda}_{,\mu} y_{\nu}^A \partial_{y_{\mu}^A} + \xi'^{\nu}_{,\lambda} \xi^{\lambda}_{,\mu} y_{\nu}^A \partial_{y_{\mu}^A} \\ &= \xi_{\xi} \xi' - i_{\mathbf{d}\xi} \xi' \mathbf{y}^A \partial_{\mathbf{y}^A} \\ &= \mathbf{w}(\xi_{\xi} \xi'), \end{aligned}$$

which proves *ii)*.

The last statement *iii)* is given by calculation

$$\begin{aligned} \llbracket \mathbf{w}(\xi), \mathbf{v}(\Lambda) \rrbracket &= \llbracket \xi, -\Lambda^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \rrbracket + \left\llbracket \xi, \frac{1}{2} \mathbf{D} \Lambda^{cd} \partial_{\mathbf{A}^{bc}} \right\rrbracket + \\ &\quad + \left\llbracket -i_{\mathbf{d}\xi} \mathbf{e}^a \partial_{\mathbf{e}^a}, -\Lambda^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \right\rrbracket + \left\llbracket -i_{\mathbf{d}\xi} \mathbf{e}^a \partial_{\mathbf{e}^a}, \frac{1}{2} \mathbf{D} \Lambda^{cd} \partial_{\mathbf{A}^{bc}} \right\rrbracket + \\ &\quad + \left\llbracket -\frac{1}{2} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \partial_{\mathbf{A}^{ab}}, -\Lambda^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \right\rrbracket + \left\llbracket -\frac{1}{2} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{D} \Lambda^{cd} \partial_{\mathbf{A}^{bc}} \right\rrbracket. \end{aligned}$$

The fourth and fifth terms are obviously vanishing. The third term is also vanishing because

$$\left\llbracket -i_{\mathbf{d}\xi} \mathbf{e}^a \partial_{\mathbf{e}^a}, -\Lambda^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \right\rrbracket = \Lambda^c_d i_{\mathbf{d}\xi} \mathbf{e}^d \partial_{\mathbf{e}^c} - i_{\mathbf{d}\xi} (\Lambda^c_d \mathbf{e}^d) \partial_{\mathbf{e}^c} = 0.$$

Calculation of the remaining terms yields expressions:

the first term

$$\left\llbracket \xi, -\Lambda^c_d \mathbf{e}^d \partial_{\mathbf{e}^c} \right\rrbracket = -\xi_{\xi} \Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a},$$

the second term

$$\left\llbracket \xi, \frac{1}{2} \mathbf{D} \Lambda^{cd} \partial_{\mathbf{A}^{bc}} \right\rrbracket = \frac{1}{2} (\xi_{\xi} - i_{\mathbf{d}\xi}) \mathbf{d} \Lambda^{ab} \partial_{\mathbf{A}^{ab}} - \xi_{\xi} \Lambda^a_{\bar{a}} \mathbf{A}^{\bar{a}b} \partial_{\mathbf{A}^{ab}}$$

and the sixth term

$$\left\llbracket -\frac{1}{2} i_{\mathbf{d}\xi} \mathbf{A}^{ab} \partial_{\mathbf{A}^{ab}}, \frac{1}{2} \mathbf{D} \Lambda^{cd} \partial_{\mathbf{A}^{bc}} \right\rrbracket = \frac{1}{2} i_{\mathbf{d}\xi} \mathbf{d} \Lambda^{ab} \partial_{\mathbf{A}^{ab}}.$$

Therefore

$$\llbracket \mathbf{w}(\xi), \mathbf{v}(\Lambda) \rrbracket = -\xi_{\xi} \Lambda^a_b \mathbf{e}^b \partial_{\mathbf{e}^a} + \frac{1}{2} \mathbf{D} \xi_{\xi} \Lambda^{ab} \partial_{\mathbf{A}^{ab}} = \mathbf{v}(\xi_{\xi} \Lambda).$$

□

Now, we want to explore transformation on \mathbf{P} given by composition $0(\Lambda) \circ \eta_{\mathbf{P}}(\xi)$. From definitions (35) and (55) we have

$$\begin{aligned} 0(\Lambda) \circ \eta_{\mathbf{P}}(\xi) : (x^{\mu}, \mathbf{e}^a, \mathbf{A}^{ab}) &= (\eta_{\mathbf{M}}^{\mu}, 0^a_b (\eta_{\mathbf{M}}^{-1})^* \mathbf{e}^b, 0^a_{\bar{a}} 0^b_{\bar{b}} (\eta_{\mathbf{M}}^{-1})^* \mathbf{A}^{\bar{a}\bar{b}} + 0^a_{\bar{a}} \eta^{\bar{a}\bar{b}} \mathbf{d} 0^b_{\bar{b}}) \\ &= (\eta_{\mathbf{M}}^{\mu}, (\eta_{\mathbf{M}}^{-1})^* ((\eta_{\mathbf{M}})^* 0^a_b \mathbf{e}^b), (\eta_{\mathbf{M}}^{-1})^* ((\eta_{\mathbf{M}})^* 0^a_{\bar{a}} (\eta_{\mathbf{M}})^* 0^b_{\bar{b}} \mathbf{A}^{\bar{a}\bar{b}} + (\eta_{\mathbf{M}})^* 0^a_{\bar{a}} \eta^{\bar{a}\bar{b}} \mathbf{d} (\eta_{\mathbf{M}})^* 0^b_{\bar{b}})) \\ &= \eta_{\mathbf{P}}(\xi) \circ 0((\eta_{\mathbf{M}})^* \Lambda) : (x^{\mu}, \mathbf{e}^a, \mathbf{A}^{ab}), \end{aligned}$$

which yields an equality

$$O(\Lambda) \circ \eta_P(\xi) = \eta_P(\xi) \circ O((\eta_M)^* \Lambda).$$

This implies that if we define G as a set of all elements of the type $O(\Lambda) \circ \eta_P(\xi)$ then (G, \circ) is a group given by semidirect product $G = SO(\mathfrak{g}, \mathbf{M}) \ltimes \mathcal{D}\text{iff}(\mathbf{M})$, where $SO(\mathfrak{g}, \mathbf{M})$ is normal subgroup of G . Since G is localizable the symmetry group is a gauge group.

Finally, we can formulate the main theorem of this subsection.

Theorem IV.1. *Group G given by semidirect product $G = SO(\mathfrak{g}, \mathbf{M}) \ltimes \mathcal{D}\text{iff}(\mathbf{M})$ is a gauge group of the Einstein-Cartan theory. Vanishing of all Noether's charges associated to the generators of G on \mathbf{P} is equivalent to the Hamilton equations of motion (39) and (40). The integral Poisson brackets of the Noether's charges $\mathbf{T}_\Sigma^\varphi(\Lambda)$ and $\mathbf{R}_\Sigma^\varphi(\xi)$ on arbitrary embedding Σ are*

i)

$$\llbracket \mathbf{t}(\Lambda), \mathbf{t}(\Lambda') \rrbracket = \mathbf{t}(\Lambda \eta \Lambda' - \Lambda' \eta \Lambda),$$

ii)

$$\llbracket \mathbf{r}(\xi), \mathbf{r}(\xi') \rrbracket = \mathbf{r}(\mathcal{L}_\xi \xi'),$$

iii)

$$\llbracket \mathbf{r}(\xi), \mathbf{t}(\Lambda) \rrbracket = \mathbf{t}(\mathcal{L}_\xi \Lambda).$$

Proof. Statements of the theorem are direct consequences of the previous two lemmata, the relation (19) and discussion in the text above. \square

V. CONCLUSION

The main goal of the series is to propose as hypothesis a new theory of covariant quantum gravity with continuous quantum geometry. The first part of the series was dealing with the covariant hamiltonian formulation of the Einstein-Cartan theory. We found out that the kinematical phase space is given as a d-jet dual $\mathbf{J}\mathbf{E}^*$ of the graded manifold of right-handed coframes over the space-time manifold \mathbf{M} with the Cartan-Poincaré form slightly modified by addition of the exact term, which is an artefact of the multisymplectic reduction since the Lagrangian of the Einstein-Cartan theory is singular. We showed that the gauge group G of the Einstein-Cartan theory is given by the semidirect product $G = SO(\mathfrak{g}, \mathbf{M}) \ltimes \mathcal{D}\text{iff}(\mathbf{M})$ of the local (proper) Lorentz group $SO(\mathfrak{g}, \mathbf{M})$ and the group of spacetime diffeomorphisms $\mathcal{D}\text{iff}(\mathbf{M})$ hence the local Poisson algebra of its generators is closed Lie algebra. Vanishing of the all Noether's charges is equivalent to equations of motion of the Einstein-Cartan theory.

These suggest that we solved the old Kuchař's problem of finding the formulation of constraints which forms closed Lie algebra, but the opposite is truth. We should take into account, that we just proved that only the local algebra is closed. In order to find an integral version of just obtained results we should, at first, introduce an instantaneous formalism what is the goal of the next part of the series.

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